Observations on Condon's Paper on the Fourier Transform

By

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Let L_2 be the class of complex-valued measurable function $f(x)$ defined over $(-\infty, +\infty)$ for which $\int_{-\infty}^{+\infty}|f(x)|^2 dx < +\infty$. Then L_2 is a Hilbert space with the inner product $(f, g) = \int_{-\infty}^{+\infty} f(x)\overline{g(x)}\,dx$. Let $\mathfrak{D}$ be the group of rotations of a plane about a fixed point. The classical Plancherel-Fourier transform is a unitary operator in L_2 and generates a cyclical group of order 4 which is isomorphic with the sub-group of $\mathfrak{D}$ through multiples of a right angle. E. U. Condon[1] proposed to find explicitly a family of unitary operators which is isomorphic with $\mathfrak{D}$ and immerses the Fourier transform. But his treatments are not rigorous, because of the introduction of an improper function $\delta(x)$ defined by the properties $\int_{-\infty}^{+\infty}\delta(x)dx = 1$ and $\delta(x) = 0$ for $x \neq 0$. Also, he makes some erroneous assertions, owing to his miscalculation.[2] Here I shall deal with the same problem and precise his results through the theory of differential set functions developed by F. Maeda. As we shall use the definitions and notations of that theory, the reader is refered to Maeda's papers[3] in this Journal.

I. Let $\beta(E)$ be a completely additive, non-negative differential set function in an abstract space Ω. We shall confine ourselves to $\mathfrak{L}_2(\beta)$, which is a Hilbert space. Let $\mathfrak{R}_\theta(E, E')$, $-\infty < \theta < +\infty$, belonging to $\mathfrak{L}_2(\beta, \beta)$.[4] If U_θ stand for operators which have $\mathfrak{R}_\theta(E, E')$ as the

(1) E. U. Condon, Proc. Nat. Acad. Sci. **23** (1937), 158–164.

(2) He obtains a double-valued representation of the group $\mathfrak{D}$, and asserts that the functional transform does not approach the identity as $\theta \to 0$; but this is not true. Cf. Condon, ibid., 163–164.

(3) e. g. F. Maeda, this Journal, 6 (1936), 19–45.

(4) F. Maeda, ibid., 31–32.