Logical Structures of Orthogonal Systems in Hilbert Space.

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In order to remove difficulties in the treatment of continuous spectrum I have introduced two kinds of orthogonal systems in Hilbert space, viz. the orthogonal system of closed linear manifolds $\{\mathfrak{M}_U\}$, and the orthogonal system of elements $\{\mathfrak{q}(U)\}$, which both have set indices U.[1] In the present paper I investigate the structures of these orthogonal systems in terms of the lattice theory.

If we consider the manifold implication as the inclusion in the definition of lattice, $\{\mathfrak{M}_U\}$ is a complemented distributive lattice. And in $\{\mathfrak{M}_U\}$ the manifold calculations obey the same laws as in the set calculations; for example,

$$\mathfrak{M}_U \geqq \mathfrak{M}_{U'} \qquad \text{when} \qquad U \geqq U',$$

$$\mathfrak{M}_U \mathfrak{M}_{U'} = \mathfrak{M}_{UU'}, \qquad \mathfrak{M}_U \oplus \mathfrak{M}_{U'} = \mathfrak{M}_{U \dotplus U'}.$$

Let $\mathfrak{f}$ and $\mathfrak{g}$ be any elements in $\mathfrak{H}$, when $(\mathfrak{f}, \mathfrak{g}) = \|\mathfrak{f}\|^2$, we write, as v. Sz. Nagy,[2] $\mathfrak{f} \prec \mathfrak{g}$. If we use this "$\prec$" as the inclusion in the lattice theory, then $\{\mathfrak{q}(U)\}$ is a complemented distributive lattice. And if we denote the meet and join of $\mathfrak{q}(U), \mathfrak{q}(U')$ by $\mathfrak{q}(U) \cdot \mathfrak{q}(U')$ and $\mathfrak{q}(U) \dotplus \mathfrak{q}(U')$ respectively, we have the following relations similar to the set calculations:

$$\mathfrak{q}(U) \succ \mathfrak{q}(U') \qquad \text{when} \qquad U \geqq U',$$

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(1) F. Maeda, this Journal, **4** (1934), 57–91; **6** (1936), 115–137; **7** (1937), 103–114, 191–213.

(2) B. v. Sz. Nagy, „Über die Gesamtheit der characteristischen Funktionen im Hilbertischen Funktionenraum", Acta Szeged, **8** (1937), 167. When $\{\mathfrak{q}(U)\}$ is complete in $\mathfrak{H}$, $\mathfrak{q}(U)$ is represented by $\sigma(EU)$, $\sigma(U)$ being $\|\mathfrak{q}(U)\|^2$. Since $\sigma(EU)$ is the integral of the characteristic function of U with respect to $\sigma(E)$, the conditions obtained by Nagy are nothing but the condition for a system of elements to be an orthogonal system of the form $\{\mathfrak{q}(U)\}$. But on the lattice theory Nagy's conditions cannot be used.