Relative Dimensionality in Operator Rings.

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In a Hilbert space, let **M** be a ring containing 1. We write $\mathfrak{M} \sim \mathfrak{N}$ $(\ldots . \mathbf{M})$ if a partially isometric operator $U \in \mathbf{M}$ exists, the initial and final sets of which are $\mathfrak{M}$ and $\mathfrak{N}$ respectively. When **M** is a factor, F. J. Murray and J. v. Neumann have proved the following comparability theorem: "If $\mathfrak{M}, \mathfrak{N} \,\eta\, \mathbf{M}$, then either $\mathfrak{M} \sim \mathfrak{N}' \subset \mathfrak{N}$ or $\mathfrak{N} \sim \mathfrak{M}' \subset \mathfrak{M}$."(1)

In the present paper I shall investigate the case where **M** is not a factor, and obtain the same results (cf. Theorems I–IV below) as those in reducible continuous geometry.

From this fact we may conjecture that with respect to dimensionality there is a lattice theory which contains both the continuous geometry and the operator rings.

1. In a Hilbert space $\mathfrak{H}$, let **M** be a ring containing 1. Denote by **E** the set of all projections E belonging to **M**. When $EF=FE=E$, we write $E \leqq F$. Let $\mathfrak{M}, \mathfrak{N}$ be the ranges of E, F respectively, then $E \leqq F$ if and only if $\mathfrak{M} \subset \mathfrak{N}$. Hence **E** is a partially ordered system with the order $\leqq$. Since $E = P_{\mathfrak{M}} \in \mathbf{E}$ if and only if $U\mathfrak{M} = \mathfrak{M}$ for every unitary $U \in \mathbf{M}'$,(2) it is evident that E is a lattice, where the join $P_{\mathfrak{M}} \vee P_{\mathfrak{N}}$ is the projection whose range is $[\mathfrak{M}, \mathfrak{N}]$, and the meet $P_{\mathfrak{M}} \wedge P_{\mathfrak{N}}$ is the projection whose range is $\mathfrak{M} \cdot \mathfrak{N}$. $E \vee F = E + F$ if and only if $EF=0$ or $FE=0$, and in this case $E \perp F$.(3) $E \wedge F = EF$ if and only if, $EF=FE$. 0 and 1 are the zero and unit elements of **E**. If $E \leqq F$, then $F-E$ belongs to **E**. And

$$E \vee (F-E) = F, \qquad E \wedge (F-E) = 0 .$$

Hence **E** is a complemented lattice. But **E** is not necessarily modular. For example, when **M** is the set of all bounded operators in $\mathfrak{H}$, then **E** is not modular.(4)

We write $\mathfrak{M} \sim \mathfrak{N}$ $(\ldots . \mathbf{M})$, and for $E = P_{\mathfrak{M}}$, $F = P_{\mathfrak{N}}$, $E \sim F$ $(\ldots . \mathbf{M})$, if a partially isometric $U \in \mathbf{M}$ exists, the initial and final sets of which are $\mathfrak{M}$

(1) Murray and v. Neumann [1], Lemma 6.2.3. The numbers in square brackets refer to the list given at the end of this paper.

(2) Murray and v. Neumann [1], 141.

(3) $E \perp F$ means that the ranges of E and F are orthogonal.

(4) Cf. G. Birkhoff and J. v. Neumann, *The Logic of Quantum Mechanics*, Annals of Math. **37** (1936), 832.