CANONICAL SUBDIRECT FACTORIZATIONS OF LATTICES

By

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G. Birkhoff has proved the following factorization theorem[1]:

Every algebra A with finitary operations can be represented as a subdirect union of subdirectly irreducible algebras. And the subdirect factorizations are closely related to the structure of the lattice[2] $\Theta(A)$ of all congruence relations on A. In particular, for a lattice L $\Theta(L)$ is pseudo-complemented. Using this fact F. Maeda [1] has introduced the canonical subdirect factorization of lattices in which the unique factorization theorem is proved.

In this paper, I show that a lattice L has the canonical subdirect factorization with subdirectly irreducible factors if and only if $\Theta(L)$ is an atomic lattice. And also I give a solution for the Birkhoff's problem 72,[3] that is, a necessary and sufficient condition such that $\Theta(L)$ is a Boolean algebra is that L has a subdirect factorization with simple factors such that the components of arbitrary two elements of L are identical except a finite number of components. I wish to thank Mr. J. Hashimoto for giving me an important suggestion to the last result. I also wish to thank Prof. F. Maeda for his kind guidance.

1. Let L be an arbitrary lattice and let $\Theta(L)$ denote the set of all congruence relations on L, then $\Theta(L)$ forms a complete lattice by defining $\theta \leqq \phi$ if and only if $a \equiv b(\theta)$ implies $a \equiv b(\phi)$. The following properties are well known:

(1) $\Theta(L)$ is an upper continuous, distributive lattice.[4]

(2) In $\Theta(L)$, every element is a meet of completely meet-irreducible elements.[5]

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1) Cf. G. Birkhoff [1] 765, [2] 92. The number in square brackets refer to the list at the end of this paper.

2) G. Birkhoff [1, p. 764] has called it the *structure lattice* of A.

3) Cf. G. Birkhoff [2] 153.

4) Cf. G. Birkhoff [2] 24. A complete lattice L is called *upper continuous* when $a_\delta \uparrow a$ implies $a_\delta \frown b \uparrow a \frown b$. When L is distributive, this is equivalent to the infinite distributive law $\bigvee(a; a \in S) \frown b = \bigvee(a \frown b; a \in S)$ for all $S \subset L$ and $b \in L$. We use 0 and 1 for the zero element and the unit element of $\Theta(L)$ respectively.

5) Cf. G. Birkhoff and O. Frink [1] 304, Theorem 7. In any complete lattice an element a which can not be a meet of elements properly containing a is called *completely meet-irreducible*.