LATTICE THEORETIC CHARACTERIZATION OF AN AFFINE GEOMETRY OF ARBITRARY DIMENSIONS

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G. Birkhoff [1][1] and K. Menger [1] have characterized lattice theoretically projective and affine geometries of finite dimensions, while a projective geometry of arbitrary dimensions, finite or infinite, has been characterized by O. Frink [1] and W. Prenowitz [1].

The purpose of this paper is to characterize the lattice of all subspaces of an affine space of arbitrary dimensions. The main theorem is as follows:

THEOREM. *An abstract lattice L is isomorphic to the lattice of all subspaces of an affine space if and only if L is relatively atomic, upper continuous lattice which is semi-modular in the sense of Wilcox and satisfies the following condition:*

Let p, q, r be independent atomic elements of L, then there exists one and only one element l such that $p < l < p \cup q \cup r$, and $l \cap (q \cup r) = 0$.

In the appendix, we shall give a proof that the axiom I_7 of Hilbert [1] p. 4 is equivalent to the transitivity of parallel lines in a 3-space.

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§ 1. A Projective Space and an Affine Space.

DEFINITION 1.1. Let G be a set of points. If for any pair of distinct points p, q of G, there exists a subset $p+q$ (called *line* of G) containing p and q, which satisfies the following conditions, then G is called a *projective space*.[2]

P.1. *Two distinct points on a line determine the line.*

P.2. *If p, q, r are points not all on the same line, and u and $v (u \neq v)$ are points such that p, q, u are on a line and p, r, v are on a line, then there is a point w such that q, r, w are on a line, and also u, v, w are on a line.*

1) The numbers in square brackets refer to the list of references at the end of the paper.

2) Cf. Birkhoff [2] 116.