JOURNAL OF SCIENCE OF THE HIROSHIMA UNIVERSITY, SER. A, VOL. 17, NO. 3, MAR., 1954

Generalization of Poincaré-Bendixson Theorem

By

Minoru URABE and Shôichirô KATSUMA

(Received October 12, 1953)

Introduction.

Given the system of the differential equations

$$(E) \qquad \frac{dx}{dt}=X(x,y), \qquad \frac{dy}{dt}=Y(x,y).$$

To this system corresponds a vector field $F=(X, Y)$ in the phase plane of the variables x and y. Then, on the existence of limit-cycles of (E), Poincaré-Bendixson theorem asserts that, if two concentric Bendixson curves C_1 and C_2 which bound a closed region R free of critical points are crossed in opposite senses by the field vectors, then there exists at least one limit-cycle lying in R. In this paper, generalizing this theorem, we shall show that, *even if the boundary curves contact in some points with the field vectors, the same conclusion is also valid.*

We assume that the boundary simply closed curves C_1 (outer) and C_2 (inner) are continuously differentiable up to the second order[1] and that the functions $X(x,y)$ and $Y(x,y)$ are continuous with their derivatives in the open region G which contains the closed region R. Then our theorem will assert that, *if R does not contain critical points and C_1 and C_2 are crossed in opposite senses or touched by the field vectors, then there exists at least one limit-cycle in R.*

§ 1. Variation of the inclinations.

In this paragraph, we denote any one of C_1 and C_2 by C. Let $x=x(s)$, $y=y(s)$ be the equations of C, where s is the length of the arc. Let the direction cosines of the tangent and the normal of C in any point $P(x,y)$ be (α,β) and (l,m) respectively. We make the convention that the positive sense of the normal is such that the tangent and the normal have the positive orientation. Then, for these direction cosines, it is evidently valid that

$$(1.1) \qquad \begin{vmatrix} \alpha & \beta \\ l & m \end{vmatrix}=+1.$$

1) When the equations of the curve are given in the form $x=x(s)$, $y=y(s)$, where s is the length of the arc, we say that the curve is continuously differentiable up to the n-th order when the functions of the right-hand sides of the equations are continuously differentiable with regard to s up to the n-th order.