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Finite-Dimensionality of Certain Banach Algebras

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The totality $\mathscr{B}$ of bounded linear operators T on a Hilbert space $\mathfrak{H}$ to itself is a Banach algebra (C*-algebra) under the norm $\| T\| = \underset{\|f\|=1}{l.u.b.} \| Tf\|$. It is known that $\mathscr{B}$ is reflexive if and only if $\mathscr{B}$ is finite-dimensional [6]. The main purpose of this paper is to show that this is also true for B*-algebras and certain other Banach *-algebras (Theorem 2). And we show that a completely continuous linear operator in $\mathfrak{H}$ is characterized as a weakly completely continuous element of the Banach algebra $\mathscr{B}$ (Theorem 4).

1. An algebra $\mathfrak{A}$ over the complex field $\boldsymbol{C}$ is called a *-algebra provided there is defined in $\mathfrak{A}$ an involution $x \rightarrow x^*$ which is a conjugate-linear anti-automorphism of period two. If $\mathfrak{A}$ is also a B-algebra, then $\mathfrak{A}$ is called a Banach *-algebra [15]. A subalgebra of a *-algebra is called a *-subalgebra provided it is closed under the involution. An element x of a *-algebra is said to be self-adjoint if $x=x^*$, normal if $xx^*=x^*x$.

Let $\mathfrak{A}$ be a *-algebra. Any commutative *-subalgebra is, by Zorn's lemma, contained in a maximal one $\mathfrak{B}$. A commutative *-subalgebra is maximal if and only if it coincides with its commutor. $\mathfrak{B}$ will be closed if $\mathfrak{A}$ is a Banach *-algebra.

LEMMA 1. *Let $\mathfrak{A}$ be a *-algebra such that every maximal commutative *-subalgebra of $\mathfrak{A}$ has a unit and no nilpotent self-adjoint elements. Then $\mathfrak{A}$ has a unit.*

PROOF. Let $\mathfrak{B}$ and $\mathfrak{B}'$ be maximal commutative *-subalgebras of $\mathfrak{A}$. Let e, e' be a unit of $\mathfrak{B}$, $\mathfrak{B}'$ respectively. They are evidently self-adjoint. Since there exist no non zero self-adjoint elements annihilating a maximal commutative *-subalgebra with a unit, hence we obtain

$$e'=e'e+ee'-ee'e. \tag{1}$$

$$e=ee'+e'e-e'ee'. \tag{2}$$

From (1) we have

$$e'=e'e'e'=2e'ee'-e'ee'ee'. \tag{3}$$

$$ee'e=2ee'ee'e-ee'ee'ee'e. \tag{4}$$

If we put $u=ee'e$, then u is self-adjoint. (4) implies $(u-u^2)^2=0$. Hence by assumption we obtain $u=u^2$. In like manner $e'ee'$ is an idempotent. Then from (3) $e'=e'ee'$, therefore from (2) $e+e'-ee'-e'e=0$, that is, $(e-e')^2=0$, which implies by assumption