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The Intersection Theorem on Noetherian Rings

By

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Introduction. The intersection theorem, due to Krull, Chevalley and Zariski, is of paramount importance in the theory of Noetherian rings. But it being stated only for rings with unit elements, we tried to formulate and prove the theorem for rings not assumed to have unit elements. Our proof is based on an investigation of relations between ideals in a ring and those in a ring obtained by adjoining a unit element to the original one. This may be blamed too long and too compliciated in order to obtain only the theorem in view. However Lemmas I* and III, which state relations between two rings, will be of some interest in themselves.

1. Theorems known. For the sake of convenience, we state the theorems due to Krull, Chevalley and Zariski, and sketch their proofs briefly.

Let $\mathfrak{o}$ be a commutative ring and form the set $\mathfrak{o}^*$ which consists of all pairs $[a, m]$ where $a \in \mathfrak{o}$ and m is an integer. $\mathfrak{o}^*$ becomes a ring if we define addition and multiplication in $\mathfrak{o}^*$ as follows:

$$[a, m]+[b, n]=[a+b, m+n],$$
$$[a, m]\cdot[b, n]=[ab+na+mb, mn].$$

We may identify $[a, 0]$, $[0, m]$ with a, m respectively in obvious reasons. Then $\mathfrak{o}^*$ is a ring with the unit element 1, and contains $\mathfrak{o}$ and the ring of integers Z as an ideal and a subring respectively. Every ideal in $\mathfrak{o}$ is an ideal in $\mathfrak{o}^*$. $\mathfrak{o}$ is a prime ideal in $\mathfrak{o}^*$, and every prime ideal in $\mathfrak{o}^*$, which contains $\mathfrak{o}$ strictly and is not equal to $\mathfrak{o}^*$, is a maximal ideal. If $\mathfrak{o}$ is Noetherian, $\mathfrak{o}^*$ is too.

(L_1) *Let* $\mathfrak{a}$ *be an ideal in a Noetherian ring and put* $\mathfrak{c}=\bigcap_{n=1}^{\infty}\mathfrak{a}^n$ *then we have* $\mathfrak{c}\mathfrak{a}=\mathfrak{c}$.

Let $\mathfrak{c}\mathfrak{a}=[\mathfrak{q}_1, \cdots, \mathfrak{q}_r]$ be a representation of $\mathfrak{c}\mathfrak{a}$ as an intersection of primary ideals and let $\mathfrak{p}_i$ be the prime divisor of $\mathfrak{q}_i$. If $\mathfrak{a} \not\leq \mathfrak{p}_i$, then $\mathfrak{c} \leq \mathfrak{q}_i$ follows from $\mathfrak{c}\mathfrak{a} \leq \mathfrak{q}_i$. If $\mathfrak{a} \leq \mathfrak{q}_i$, then $\mathfrak{c} \leq \mathfrak{a}^{\rho_i} \leq \mathfrak{p}_i^{\rho_i} \leq \mathfrak{q}_i$, ρ_i being the exponent of $\mathfrak{q}_i$. After all we have always $\mathfrak{c} \leq \mathfrak{q}_i$.

(L_2) *Let* $\mathfrak{c}$, $\mathfrak{a}$ *be ideals in a commutative ring* $\mathfrak{o}$ *such that* $\mathfrak{c}\mathfrak{a}=\mathfrak{c}$ *and suppose* $\mathfrak{c}$ *has a finite base. Then there exists an element a in* $\mathfrak{c}$ *such that c·a=c for any element c in* $\mathfrak{c}$.[1)]

1) S. Mori, *Ueber Productzerlegung der Ideals.* This Journal 2 (1932) Satz 1, p. 1.