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Orthocomplemented Lattices Satisfying the Exchange Axiom

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The set of all closed linear manifolds of a (not necessarily separable) Hilbet space is a (I) *complete*, (II) *relatively atomic*, and (IV) *orthocomplemented* lattice satisfying (III) the *exchange axiom* of MacLane [1][1], together with the following condition:

(V) *If* b, c *are orthogonal elements, then it holds* $(b,c)M$, i.e., $a\leqq c$ *implies* $(a\smile b)\frown c=a\smile(b\frown c)$.

The purpose of this paper is to study an abstract lattice L satisfying these five conditions (I)-(V). The main results are as follows[2]:

(1) Any element a of L has an orthogonal basis, that is, there exists an orthogonal system of points whose join is equals to a.

(2) If P, Q are both orthogonal bases of an element, then P and Q have the same cardinal numbers provided that L satisfies furthermore an "counrability condition of dependence".[3]

(3) Any quotient lattice of L has the same properties as L.

(4) L is a direct sum of irreducible sublattices.

(5) Projections and permutability of elements are defined and their interrelation is investigated.

§1. The lattice of closed linear manifolds of a Hibert space.

Let $\mathfrak{H}$ be a (not necessarily separable) Hilbert space, and let the set of all closed linear manifolds of $\mathfrak{H}$ be denoted by L. It is well known that L is a complete, relatively atomic, and orthocomplemented lattice, partially ordered by set-inclusion. Croisot [1] has shown that L satisfies the exchange axiom of MacLane[4]. Hence we have the following:

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1) The numbers in square brackets refer to the list of references at the end of the paper.

2) An *exchange lattice* of MacLane [1], which is equivalent to a "*matroid lattice*" in the terminology of F. Maeda [3], is a complete, relatively atomic lattice satisfying the exchange axiom together with the "finiteness condition of dependence". MacLane [1] has shown that analogous theorems to (1)-(3) above are valid in any exchange lattice (cf. ibid. 458, Theorem 3, 4 and 6); (4) has been shown for an exchange lattice by U. Sasaki and S. Fujiwara [1] 188, Theorem 4.

3) Cf. the condition (VI) in Theorem 2.2 below.

4) Cf. Croisot [1] 259, Lemma 1 and 261, Lemma 3 which are respectively (η'') and (ξ'') in Remark (2) below.