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A Note on Semi-Local Rings

By

Michio YOSHIDA and Motoyoshi SAKUMA

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The object of this note is to give the complete information on the indecomposable components of the completions of semi-local rings.

Let $\mathfrak{o}$ be a commutative ring, and let $\mathfrak{m}$ be an ideal in $\mathfrak{o}$ such that $\bigcap_{n=1}^{\infty}\mathfrak{m}^n=(0)$. The metrisable, uniform structure, defined in $\mathfrak{o}$ by adopting the set $\{\mathfrak{m}^n;\ n=1,2,\cdots\}$ as a fundamental system of neighbourhoods of zero, shall be called an $\mathfrak{m}$-adic topology. If we give this topology to $\mathfrak{o}$, it becomes a topological ring and we shall call thus topologized ring an $\mathfrak{m}$-adic ring.

The completion of an $\mathfrak{m}$-adic ring $\mathfrak{o}$ shall be called an $\mathfrak{m}$-adic completion of $\mathfrak{o}$, and shall be denoted by $\bar{\mathfrak{o}}$. If we denote by $\overline{\mathfrak{m}^\sigma}$ the adherence of $\mathfrak{m}^\sigma$ in $\bar{\mathfrak{o}}$, the set $\{\overline{\mathfrak{m}^\sigma};\ \sigma=1,2,\cdots\}$ is a fundamental system of neighbourhoods of zero in $\bar{\mathfrak{o}}$. If $\mathfrak{m}$ has a finite base, then we have $\overline{\mathfrak{m}^\sigma}=\bar{\mathfrak{m}}^\sigma$, and $\bar{\mathfrak{o}}$ is an $\bar{\mathfrak{m}}$-adic ring. If moreover $\mathfrak{o}$ has a unit element, then $\bar{\mathfrak{m}}=\mathfrak{m}\bar{\mathfrak{o}}$. If $\mathfrak{o}$ is a Noetherian ring (that is, a commutative ring with the maximal condition for ideals) with a unit element, so is $\bar{\mathfrak{o}}$ too.

DEFINITION (D). Let $\mathfrak{m}$, $\mathfrak{m}_1$ be ideals in $\mathfrak{o}$, such that $\mathfrak{m}\subseteq\mathfrak{m}_1$, and $\bigcap\mathfrak{m}^n=(0)$. Set $\bigcap\mathfrak{m}_1^n=\mathfrak{m}_1^\infty$, $\mathfrak{o}/\mathfrak{m}_1^\infty=\mathfrak{o}_1'$, $\mathfrak{m}_1/\mathfrak{m}_1^\infty=\mathfrak{m}_1'$, and let $\bar{\mathfrak{o}}$, $\bar{\mathfrak{o}}_1$ be the $\mathfrak{m}$-adic and the $\mathfrak{m}_1'$-adic completion of $\mathfrak{o}$ and $\mathfrak{o}_1'$ respectively. Let x^* be any element in $\bar{\mathfrak{o}}$, and let $x^*=\lim x_n$ $(x_n\in\mathfrak{o})$. Then if we denote by x'_n the residue class modulo $\mathfrak{m}_1^\infty$ which contains x_n, $\{x'_n\}$ is a Cauchy sequence in the $\mathfrak{m}_1'$-adic ring $\mathfrak{o}_1'$. If we denote by x^{**} the limit of $\{x'_n\}$ in $\bar{\mathfrak{o}}_1$, then the mapping $\tau_1:\ x^*\to x^{**}$ is clearly a homomorphism (that is, a continuous ring-homomorphism) of $\bar{\mathfrak{o}}$ into $\bar{\mathfrak{o}}_1$. This shall be called the canonical homomorphism of $\bar{\mathfrak{o}}$ into $\bar{\mathfrak{o}}_1$.

Now, let $\mathfrak{m}_i$ $(i=1,2,\cdots,r)$ be ideals in $\mathfrak{o}$ such that $\mathfrak{m}\subseteq\mathfrak{m}_i$, and define $\bar{\mathfrak{o}}_i$, τ_i similarly as $\mathfrak{o}_1'$, τ_1. Then the mapping τ of $\bar{\mathfrak{o}}$ into the direct sum $\tilde{\mathfrak{o}}$ of $\bar{\mathfrak{o}}_1,\cdots,\bar{\mathfrak{o}}_r$ defined by setting $\tau x^*=\tau_1 x^*+\cdots+\tau_r x^*$, shall be called the canonical homomorphism of $\bar{\mathfrak{o}}$ into $\tilde{\mathfrak{o}}$.

THEOREM I. *Let $\mathfrak{o}$ be a commutative ring with a unit element, and let $\mathfrak{m}$ be an ideal in $\mathfrak{o}$ such that $\bigcap\mathfrak{m}^n=(0)$. Suppose that*

$$\mathfrak{m}=\mathfrak{m}_1\cap\cdots\cdots\cap\mathfrak{m}_r,$$

where $\mathfrak{m}_i$ $(i=1,2,\cdots,r)$ are ideals in $\mathfrak{o}$, such that $(\mathfrak{m}_i,\mathfrak{m}_j)=(1)$ for $i\neq j$. Then (with the same notations as in (D)), the canonical homomorphism τ is an isomorphism of $\bar{\mathfrak{o}}$ onto $\tilde{\mathfrak{o}}$.

PROOF. We shall first prove that τ is a mapping on $\tilde{\mathfrak{o}}$. Let x_i^* be any element in $\bar{\mathfrak{o}}_i$, and let $x_i^*=\lim x_\nu''$ $(x_\nu''\in\mathfrak{o}_i')$. Let x'_ν be any element in the residue class x_ν'', then there exists an element x_ν in $\mathfrak{o}$ such that