JOURNAL OF SCIENCE OF THE HIROSHIMA UNIVERSITY, SER. A. VOL. 18, No. 3, MARCH, 1955

A Non-Commutative Theory of Integration for Operators

By

Tôzirô OGASAWARA and Kyôichi YOSHINAGA

(Received Sept. 20, 1954)

The so-called "non-commutative theory" of integration for rings of operators on Hilbert spaces has been much developed by Segal [**26**] and Dixmier [**9**], independently. The former's theory is a theory of integrals (or traces) for certain (unbounded) "measurable operators", analogous to measurable functions in the classical theory of integrations over abstract measure spaces. His idea of the "measurable operators" originates from the works of Murray and v. Neumann ([**18**], Chap. 16) for factors of type II, and of Dye [**11**] for finite rings. The latter's theory is a theory of integrals as linear forms. For both theories the rings may be assumed to be semi-finite without loss of generality. A ring $\mathbb{M}$ of operators is called semi-finite [**15**] provided every non-zero projection $\in \mathbb{M}$ contains a non-zero finite projection $\in \mathbb{M}$. Let $\mathbb{M}$ and $\mathbb{N}$ be $*$-isomorphic rings of operators, and let m and μ be regular gages of $\mathbb{M}$ and $\mathbb{N}$ respectively such that m and μ correspond by means of the above $*$-isomorphism. If we stand on the view-point of Dixmer [**9**], the measurable integrable operators with respect to m and μ must correspond $*$-isomorphically. We show (Theorem 1) that if $\mathbb{M}$ is $*$-isomorphic with $\mathbb{N}$ by means of a mapping θ, then θ is uniquely extended to a $*$-isomorphic mapping between measurable operators with respect to $\mathbb{M}$ and $\mathbb{N}$. To develop the theory of Segal [**26**] for a given ring $\mathbb{M}$ it seems, therefore, preferable to take an appropriate ring $\mathbb{N}$ $*$-isomorphic with $\mathbb{M}$ and to develop the theory for $\mathbb{N}$ instead of $\mathbb{M}$ and then to transfer it to that for $\mathbb{M}$, if such a process is more suitable. It is known that every semi-finite ring $\mathbb{M}$ is $*$-isomorphic with the left ring $\mathbb{L}$ of an H-system $\mathbf{H}$, and the regular gage of $\mathbb{M}$ in question corresponds to the canonical gage μ of $\mathbf{H}$. Left multiplication operators L_x, $x \in \mathbf{H}$ form a Hilbert space when the inner product $\langle L_x, L_y \rangle$ is defined by $\langle L_x, L_y \rangle = \langle x, y \rangle$. The set $\mathfrak{L}_2$ of all L_x is the set of square integrable measurable operators with respect to μ. Thus in $\mathbf{H}$ the square integrable measurable operators are given *a priori*. We define that $T = L_x \cdot L_y$ is integrable with respect to μ and define its integral $\mu(T)$ by $\langle L_x, L_{y^*} \rangle$. Let $\mathfrak{L}_1$ be