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A Theorem on Operator Algebras

By

Tôzirô OGASAWARA

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Let $\boldsymbol{A}$ be a B^*–algebra, that is, a Banach $*$–algebra with the property $\|A^*A\| = \|A\|^2$ for every $A \in \boldsymbol{A}$. Such an algebra is $*$–isomorphically representable as a uniformly closed algebra of operators on a Hilbert space $\mathfrak{H}$. In the sequel we assume that $\boldsymbol{A}$ is represented such an algebra of operetors on $\mathfrak{H}$. It is the purpose of this paper to prove the following theorem [1]:

THEOREM. *Let $\boldsymbol{A}$ be a B^*–algebra. Then for $A, B \in \boldsymbol{A}^+$*

(a) *if $A \geq B$, then $A^{\frac{1}{2}} \geq B^{\frac{1}{2}}$;*

(b) *if $A \geq B$ implies always $A^2 \geq B^2$, then $\boldsymbol{A}$ is commutative.*

1. Proof of (a). First consider the case that $\mathfrak{H}$ is finite-dimensional. Suppose the contrary. Let $Tr(C)$ stand for an ordinary trace of operators C on $\mathfrak{H}$. It is a positive linear functional and has the property that $Tr(CD) \geq 0$ for $C \geq 0$ and $D \geq 0$. Put $S = A^{\frac{1}{2}}$ and $T = B^{\frac{1}{2}}$. Owing to the spectral resolution of $T-S$ there exists a non-zero projection P such that $P(T-S) = (T-S)P \geq \delta P > 0$ for a positive number δ. Then $Tr(P(T-S)(T+S)) \geq \delta Tr(P(T+S)) \geq 0$. On the other hand $Tr(P(T-S)(T+S)) = \frac{1}{2}\{Tr(P(T-S)(T+S)) + Tr((T+S)(T-S)P)\} = \frac{1}{2}\{Tr(P(T-S)(T+S)) + Tr(P(T+S)(T-S))\} = Tr(P(B-A)) \leq 0$. From these inequalities we have $Tr(P(T+S)) = 0$ and therefore $P(T+S)P = 0$, which entails that $PTP = PSP = 0$. Then $P(T-S) = P(T-S)P = PTS - PSP = 0$. It contradicts $P(T-S) \geq \delta P > 0$.

Now we consider the general case. Without loss of generality we may assume that $\boldsymbol{A}$ is the algebra $\mathscr{B}$ of operators on $\mathfrak{H}$. For each finite-dimensional projection P_δ we designate by A_δ the greatest positive operator $\leq A$ such that $A_\delta P_\delta = P_\delta A_\delta = A_\delta$. Such an A_δ is determined by $\langle A_\delta f, f \rangle = \underset{P_\delta g = P_\delta f}{\text{g. l. b.}} \langle Ag, g \rangle$ (cf. [**1**]). $\{A_\delta\}$ is a directed set by the ordering "$\geq$" of operator algebras and it is easy to see that $\{A_\delta\}$ converges to A in the strong topology (cf. [**1**]). $P_\delta \geq P_{\delta'}$ entails $A_\delta \geq A_{\delta'}$ and therefore $A_\delta^{\frac{1}{2}} \geq A_{\delta'}^{\frac{1}{2}}$ by the above discussion. Let T be the strong limit of the directed

1) Added in proof. (a) follows as a special case from a theorem due to E. Heinz (Math. Ann. 123 (1951), 415-438, §1 Satz 2). Cf. also T. Kato, Math. Ann. 125 (1952/53), 208-212, Theorem 2.