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A Note on the Commutativity of Certain Rings

By

Fujio Mitsudo

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In his recent paper Herstein [1] proved the following theorem: Let R be a ring with center Z. Suppose that every $x \in R$ satisfies $x^{n(x)} - x \in Z$, where $n(x) > 1$ is an integer depending on x, then R is commutative. This theorem is a generalization of a well-known theorem of Jacobson [2]: If in a ring R every $x \in R$ satisfies $x^n - x = 0$ with $n > 1$ depending on x, then R is commutative.

Throughout this note R denotes a ring with the identity element, Z is the center of R and every element of R satisfies $x^{n(x)} - h(x) \cdot x \in Z$ where $n = n(x) > 1$ is an integer bounded for all x, $h = h(x)$ is regular in Z for all $x \in R$ and the set H of elements $h(x)$ for all x is finite.

In this note, we shall prove the following theorem using Herstein's process and results [1].

Theorem. *Let R be a ring with the identity element in which every element x satisfies $x^n - hx \in Z$, where n, h depend on x, $n(>1)$ is an integer but bounded for all x, and $h \in H$ for all x, H being a finite set of regular elements contained in Z. Then R is commutative.*

In this theorem we assumed that the set H is finite. For in the division ring Q of all quaternions over the real number field every element $x = \alpha + \beta i + \gamma j + \delta k$ satisfies $x^2 - 2\alpha x \in Z$, where Z is the real number field, and when $\alpha = 0$, $x \neq 0$, x satisfies $x^3 - hx \in Z$ where $h = -(\beta^2 + \gamma^2 + \delta^2)$, and when $x = 0$, x satisfies $x^2 - x \in Z$. Therefore the ring Q, which is not commutative, does not satisfy the condition that H is finite, though it satisfies all the other conditions in the theorem.

Since we assumed that R has identity element and h is regular all the results of Herstein [1] can be obtained by a slight modification except for the division ring case and the final step of the proof of the main theorem. The proof can be divided into two cases: semi-simple ring case and general case. The first case can be reduced to the division ring case and the second can be reduced to the subdirectly irreducible case.

Division ring case. Let R be a division ring. In this case the center Z is a field.