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On the Definition of Convolutions for Distributions

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The main purpose of this paper is to show the equivalence of the definitions of convolutions available in the theory of distributions.

Let S and T be two distributions on R^n, n-dimensional Euclidean space. L. Schwartz ([12], exposé 21) defined the convolution $S*T$ by the relation

$$<S*T, \varphi>=\iint(S_x\otimes T_y)\varphi(x+y)dx\,dy \quad \text{for any} \quad \varphi\in(\mathscr{D}),$$

if the following condition is satisfied:

$$(*) \qquad (S_x\otimes T_y)\varphi(\hat{x}+\hat{y})\in(\mathscr{D}'_{L^1}) \quad \text{for any} \quad \varphi\in(\mathscr{D}).$$

In his lecture notes [4], C. Chevalley gave two definitions of convolutions. His first definition (in more precise form) is: $S*T$ is defined as

$$(1) \qquad \int_{R^n} S(y)T(\hat{x}-y)\,dy,$$

when this makes sense. This last phrase is interpreted as in the case of integration of vector-valued functions, that is, (1) has the meanings if and only if

$$(\bar{*})' \qquad S(\check{T}*\varphi)\in(\mathscr{D}'_{L^1}) \quad \text{for any} \quad \varphi\in(\mathscr{D}).$$

And

$$<\int_{R^n} S(y)T(\hat{x}-y)dy, \varphi>=\int_{R^n} S(y)(\check{T}*\varphi)(y)dy.$$

Then in the terminology of L. Schwartz ([14], p. 130), the definition is equivalent to saying that $S*T$ is the integral (1) when the integrand $S(y)T(x-y)$ is partially summable with respect to y. The second definition (generalized convolution in his sense, [4], p. 112) is:

$S*T$ is defined when the condition

$$(\bar{*}) \qquad (S*\varphi)(\check{T}*\psi)\in L^1 \quad \text{for any} \quad \varphi,\psi\in(\mathscr{D})$$

is satisfied, and $S*T$ is given by

$$<(S*T)*\varphi, \psi>=\int_{R^n}(S*\varphi)(x)(\check{T}*\psi)(x)dx.$$

In sec. 3, we show that these definitions of convolutions are equivalent, and furthermore that it remains valid that the definitions obtained by replacing $(\mathscr{D})$ by $(\mathscr{S})$ in the above discussions are also equivalent. After Hirata and Ogata [8] we say that the $(\mathscr{S}')$-convolution $S*T$ is defined when