J. Sci. Hiroshima Univ. Ser. A
Vol. 23, No. 1, April, 1959

On the Theory of Multiplicities in Finite Modules over Semi-Local Rings

Motoyoshi Sakuma

(Received January 20, 1959)

The theory of multiplicities in semi-local rings was originated by Chevalley (in geometric local rings) and continued mainly by Samuel [8] and Nagata [6]. Because of the fact that they used high techniques of the theory of local rings, especially the structure theorem of complete local rings, it is desirable to construct the theory in a more elementary and simpler way.

Godement has shown in [2] that the notion of Hilbert characteristic functions in semi-local rings can be naturally extended to that in finite modules over these rings and Serre continued the study by the homological method [14]. And recently Lech has succeeded in proving the associative formula for multiplicities without using the structure theorem [5]. We develop here systematically, following closely [14] and [5], the full theory of multiplicities in finite modules over semi-local rings and simplify their proofs in some points.

In §1 we generalize the intersection theorem of Zariski to that in finite modules. Some remarks on finite modules are stated in §2. Then, in §3, we shall generalize the theorem of Lech concerning an expression for the multiplicity of a primary ideal, which is generated by a system of parameters, in a local ring to that of a defining ideal in a finite module over a semi-local ring. In the main §4, we shall give fundamental theorems on multiplicities. Finally in §5, we treat the complete tensor product of modules.

Conventions and terminology

i) Unless stated otherwise, we assume throughout this paper that all rings are commutative Noetherian rings with identity elements and that all modules are unitary and finitely generated, therefore Noetherian.

ii) Let A be a ring. We denote by rank A the maximal number n such that there exists a chain of prime ideals of A

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$$

where each inclusion is strict. For an ideal $\mathfrak{a}$ in A, we define corank $\mathfrak{a} =$ rank $A/\mathfrak{a}$.

iii) Let E be a finite A-module. For an ideal $\mathfrak{a}$ of A and a submodule F