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On the Exchange Formula for Distributions

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Let R^n denote the n-dimensional Eulcidean space with points $x=(x_1, x_2, \cdots, x_n)$. The theory of Fourier transform of temperate distributions (elements of the space $(\mathcal{S}')$) on R^n has been developed by L. Schwartz [3]. The Fourier transform of any temperate distribution is defined as another temperate distribution. For any distribution $S\in(\mathcal{S}')$ the Fourier and inverse Fourier transforms of S are denoted by $\mathcal{F}(S)$ and $\overline{\mathcal{F}}(S)$ respectively. Let $\{\rho_k\}$ and $\{\rho'_k\}$ be any sequences of regularizations and let $r_k=\overline{\mathcal{F}}(\rho_k)$. Now let S and T be any two distributions of $(\mathcal{S}')$ such that $S*T$ is defined and belongs to $(\mathcal{S}')$. r_kT is an element of $(\mathcal{O}'_c)$ since r_k belongs to $(\mathcal{S})$. Owing to the basic exchange formula due to L. Schwartz [3], we have $\mathcal{F}(S*(r_kT)) =\mathcal{F}(S)\cdot(\mathcal{F}(T)*\rho_k)$. It follows then as noted by R. E. Edwards [2] that the exchange formula

$$\mathcal{F}(S*T)=\mathcal{F}(S)\cdot\mathcal{F}(T) \tag{1}$$

holds under the following conditions:

(α) $S*(r_kT)$ converges to $S*T$ in $(\mathcal{S}')$ as $k\to\infty$

(β) $\mathcal{F}(S)\cdot\mathcal{F}(T)$ is defined and $\mathcal{F}(S)\cdot(\mathcal{F}(T)*\rho_k)$ converges to $\mathcal{F}(S)\cdot\mathcal{F}(T)$ in $(\mathcal{D}')$ as $k\to\infty$.

Our present purpose of this paper is to eliminate, in a certain sense, the two conditions (α) and (β) in the above statement. To this end we first introduce the concept of $(\mathcal{S}')$-convolution $S*T$ (§1). We show that if $S*T$ is defined, then it belongs to $(\mathcal{S}')$ and coincides with the ordinary convolution in the sense of C. Chevalley [1]. Secondly we propose to define the muliplicative product $A\cdot B$ of two distributions A, B as the common limit of sequence $(A*\rho_k)B$ and $A(B*\rho'_k)$ in $(\mathcal{D}')$ as $k\to\infty$ provided these limits exist and coincide. In §2 we show that if, for any two distributions $S, T\in(\mathcal{S}')$, the $(\mathcal{S}')$-convolution $S \circledast T$ is defined, then the exchange formula (1) holds.

Concerning distributions, we adopt the notations of L. Shwartz [3] unless otherwise specifically mentioned.

1. Let S and T be any two distributions. Following C. Chevalley [1] we say that the convolution $S*T$ is defined if the following condition is satisfied:

$(*)$ $(S*\varphi)\cdot(\check{T}*\psi)$ belongs to L for any $\varphi, \psi\in(\mathcal{D})$,