JOURNAL OF SCIENCE OF THE HIROSHIMA UNIVERSITY, SER. A, VOL. 21, No. 3, JANUARY, 1958

On Geodesic Subspaces of Group Spaces

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(Received Sept. 26, 1957)

Introduction

The purpose of this paper is to investigate the properties of the geodesic subspaces of group spaces, mainly, by means of the exponential mapping of the Lie algebra into the Lie group. Let G be a connected real or complex Lie group with the Lie algebra $\mathfrak{G}$, and let S be a geodesic subspace (cf. §2 for definition) of the group space G with the associated linear subspace $\mathfrak{S}_x(x \in S)$ of $\mathfrak{G}$, then $\mathfrak{S}_x(x \in S)$ are all Lie triple systems (Lemma 2). In §1 we shall state the fundamental concepts which are used in the following sections. In §2 we shall define the geodesic subspaces of a group space and consider the linear subspaces $\mathfrak{S}_x$ and the relations among them. The results obtained in §2 are as follows:

1. If there exists a geodesic arc on $S: y(t)=x \exp tA$ having the end points x and y, then it holds

$$\mathfrak{S}_y=\exp\left(-\tfrac{1}{2}\operatorname{ad} A\right)\cdot\mathfrak{S}_x.$$

2. Under the same assumption as in 1, if A is a regular element of $\mathfrak{S}_x$ (cf. §1 for definition), then the intersection $\mathfrak{S}_x \cap \mathfrak{S}_y$ of $\mathfrak{S}_x$ and $\mathfrak{S}_y$ is given as

$$\mathfrak{S}_x \cap \mathfrak{S}_y=\{Y;\ Y \in \mathfrak{S}_x,\ \operatorname{ad} A\cdot Y \in \mathfrak{S}_x\}.$$

In §3, we shall consider the analogous property (Lemma 5) for the geodesic subspaces of Schreier's Theorem in the theory of topological groups, and the results obtained by means of this property are follows:

3. For any elements x and y of S, $\mathfrak{S}_y$ is transferred from $\mathfrak{S}_x$ by an inner automorphism of $\mathfrak{G}$.

4. For any elements x and y of S, there exists an analytic curve in G through x and y, which lies on $x \exp \mathfrak{S}_x$ in a sufficiently small neighborhood of x in G.

From the result 4, by means of the method of Schröder ([8])[1] the following result is obtained.

5. Let $\mathfrak{S}$ be a complex geodesic subspace of a complex linear Lie group, then we have $S \subset x\,\overline{\exp \mathfrak{S}_x}$, where $\overline{\exp \mathfrak{S}_x}$ denotes the closure of $\exp \mathfrak{S}_x$ in G.

1) Numbers in brackets refer to the references at the end of the paper.