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WAFOM over abelian groups for quasi-Monte Carlo point sets

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ABSTRACT. In this paper, we study quasi-Monte Carlo (QMC) rules for numerical integration. J. Dick proved a Koksma-Hlawka type inequality for α-smooth integrands and gave an explicit construction of QMC rules achieving the optimal rate of convergence in that function class. From this inequality, Matsumoto, Saito and Matoba introduced the Walsh figure of merit (WAFOM) $\mathrm{WF}(P)$ for an $\mathbf{F}_2$-digital net P as a quickly computable quality criterion for P as a QMC point set. The key ingredient for obtaining WAFOM is the Dick weight, a generalization of the Hamming weight and the Niederreiter-Rosenbloom-Tsfasman (NRT) weight.

We extend the notions of the Dick weight and WAFOM over a general finite abelian group G, and show that this version of WAFOM satisfies Koksma-Hlawka type inequality when G is cyclic. We give a MacWilliams-type identity on weight enumerator polynomials for the Dick weight, by which we can compute the minimum Dick weight as well as WAFOM. We give a lower bound on WAFOM of order $N^{-C_G'(\log N)/s}$ and an upper bound on lowest WAFOM of order $N^{-C_G(\log N)/s}$ for given (G, N, s) if $(\log N)/s$ is sufficiently large, where C_G' and C_G are constants depending only on the cardinality of G and N is the cardinality of quadrature rules in $[0, 1)^s$. These bounds generalize the bounds given by Yoshiki and others given for $G = \mathbf{F}_2$.

1. Introduction

Quasi-Monte Carlo (QMC) integration is a method for numerical integration using the average of function evaluations as an approximation of the true integration value. In QMC integration, sample points are chosen deterministically, while in Monte-Carlo integration they are chosen randomly. Thus, how to construct point sets has been a major concern in QMC theory. One of the known good construction frameworks is digital nets, which is based on linear algebra over finite fields (or more generally over finite rings).

A strong analogy between coding theory and QMC point sets is well known (see, e.g., [2, 13, 17]). In coding theory, the minimum Hamming weight is used for a criterion for linear codes. Analogically, Niederreiter-

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