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Existence, local uniqueness and asymptotic approximation of spike solutions to singularly perturbed elliptic problems

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ABSTRACT. This article is concerned with general singularly perturbed second order semilinear elliptic equations on bounded domains $\Omega \subset \mathbf{R}^n$ with nonlinear natural boundary conditions. The equations are not necessarily of variational type. We describe an algorithm to construct sequences of approximate *spike solutions*, prove existence and local uniqueness of exact spike solutions close to the approximate ones (using an Implicit Function Theorem type result), and estimate the distance between the approximate and the exact solutions. Here *spike solution* means that there exists a point in Ω such that the solution has a spike-like shape in a vicinity of such point and that the solution is approximately zero away from this point. The spike shape is not radially symmetric in general and may change sign.

1. Introduction

The aim of this paper is to study the existence, local uniqueness and asymptotic behaviour for $\varepsilon \to 0$ of spike solutions to singularly perturbed elliptic boundary value problems of the type

$$\begin{cases} \varepsilon^2 \left(\sum\limits_{i,j=1}^{n} \partial_{x_i}(a_{ij}(x)\partial_{x_j}u) + \sum\limits_{i=1}^{n} b_i(x)\partial_{x_i}u \right) = f(x,u,\varepsilon), & x \in \Omega, \\ \sum\limits_{i,j=1}^{n} a_{ij}(x)\nu_i(x)\partial_{x_j}u = g(x,u,\varepsilon), & x \in \partial\Omega. \end{cases} \tag{1.1}$$

Here $\varepsilon > 0$ is a small parameter, $\Omega \subset \mathbf{R}^n$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$, and ν_i are the components of the unit outer normal at $\partial\Omega$. The coefficients $a_{ij}, b_i : \overline{\Omega} \to \mathbf{R}$, and the right-hand sides $f : \overline{\Omega} \times \mathbf{R} \times [0,1] \to \mathbf{R}$ and $g : \partial\Omega \times \mathbf{R} \times [0,1] \to \mathbf{R}$ are supposed to be sufficiently smooth.

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