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# An exponential sum and higher-codimensional subvarieties of projective spaces over finite fields

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Abstract. A general method to express in terms of Gauss sums the number of rational points of general subschemes of projective schemes over finite fields is applied to the image of the triple embedding $\mathbf{P}^1 \hookrightarrow \mathbf{P}^3$. As a consequence, we obtain a non-trivial description of the value of a Kloosterman-sum-like exponential sum.

## Introduction

Based on a calculation concerning the diagonal hypersurfaces and Jacobi sums, André Weil [12] observed that the number of rational points of algebraic varieties over finite fields is a highly geometric information. His celebrated Weil conjecture is eventually proved by Pierre Deligne [1] with Grothendieck's theory of étale cohomology.

Another, and the first, proof by Bernard Dwork [3] of the rationality of zeta function, a part of Weil conjecture, more directly concerns exponential sums. In the very beginning of his proof, he reduces the rationality of the zeta function of algebraic varieties to that of hypersurfaces of $(\mathbf{G}_{\mathrm{m}})^n$, and writes the number of rational points of the hypersurface by using exponential sums.

Besides such general theories, expressing the number of rational points using exponential sums has proved to be effective also in studying concrete projective hypersurfaces; the study of Dwork families, for example by Neal Koblitz [9], is one of the most outstanding success of this strategy. It seems therefore natural to expect the efficacy of this tactic in studying projective varieties of higher codimension. This point of view, however, does not seem to be taken note of enough in the study of concrete algebraic varieties of higher codimension.

In the former half of this article, we explain the general method for obtaining a formula for the number of rational points of projective varieties in terms of Gauss sums. Then, in the latter half, we apply the method to get an expression for the number of rational points of the image $V$ of the

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