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On modified singular integrals

Dedicated to Prof. Masanori Kishi on the occasion of his 60th birthday

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1. Introduction

Let R^n be the n-dimensional Euclidean space. E. M. Stein [5] gave a weighted norm inequality for singular integrals on R^n as follows (see also C. Sadosky [4; Theorem 6.1]):

THEOREM A. *Let $\Omega(x)$ be a homogeneous function of degree $-n$ on R^n, and suppose that $\Omega(x)$ satisfies the cancellation property*

$$\int_S \Omega(x)\,d\sigma(x) = 0, \tag{1.1}$$

where $d\sigma$ is the induced Euclidean measure on the unit sphere S, and $\Omega(x)$ is bounded on S. Let $Tf(x)$ denote the corresponding singular integral:

$$Tf(x) = \lim_{\varepsilon\to 0}\int_{|x-y|\geq\varepsilon} \Omega(x-y)f(y)\,dy.$$

Then

$$\left(\int |Tf(x)|^p |x|^{-rp}\,dx\right)^{1/p} \leq C\left(\int |f(y)|^p |y|^{-rp}\,dy\right)^{1/p}, \tag{1.2}$$

provided that $1 < p < \infty$ and $-n/p' < r < n/p$ where $(1/p) + (1/p') = 1$.

For the ordinary singular integrals the above restriction of r is necessary. Indeed, when $r \geq n/p$, for $f(y) = (1 + \log|y|)^{-1}$, $|y| \geq 1$, we see $\int |f(y)|^p |y|^{-rp} dy < \infty$ and $\int_{|x-y|\geq\varepsilon} |\Omega(x-y)f(y)|\,dy = \infty$, so (1.2) fails. When $r \leq -n/p'$, for $f(y) = (1 - \log|y|)^{-1}|y|^{-\beta}$, $|y| \leq 1$, (1.2) does not hold with $n \leq \beta \leq (n/p) - r$. The purposes of this paper are to introduce modified singular integrals and give integral estimates similar to (1.2) which holds for all $r > -n/p'$ such that $r - (n/p) \neq a$ nonnegative integer.

Let $\Omega(x)$ be a homogeneous function of degree $-n$, and suppose that $\Omega(x)$ satisfies (1.1) and $\Omega(x) \in C^\infty(R^n - \{0\})$. For an integer $k \geq -1$ we set