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Note on Formal Lie Groups (II)

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1. Let K be an algebraically closed field. For any algebraic subgroup of the general linear group $GL(n, K)$ we can associate a formal Lie group, and for any subgroup G of the formal Lie group $GL^*(n, K)$ associated with $GL(n, K)$ the algebraic hull $\mathfrak{A}(G)$ can be defined in $GL(n, K)$. On the base of such connection with algebraic linear groups, non-commutative formal Lie groups were investigated in [3] by making use of the properties of algebraic linear groups in [1]. In [4], we settled some questions raised in [3] on maximal solvable subgroups, maximal tori etc. of a subgroup of $GL^*(n, K)$.

The purpose of this note is to show some properties of formal Lie groups which follow from the results in [4].

The following theorem was proved by J. Dieudonné in [3]: In order that a formal Lie group G over an algebraically closed field K of characteristic $p>0$ be nilpotent, it is necessary and sufficient that it contain a unique maximal torus. We shall give another condition for G to be nilpotent and give another proof of the sufficiency part of the theorem by using [4, Th. 2], which allows us to make use of the corresponding theorem of algebraic linear groups. We shall also show some properties of maximal unipotent subgroups of a subgroup G of $GL^*(n, K)$. E.g., if a maximal torus and a maximal unipotent subgroup are associated with algebraic subgroups of $GL(n, K)$, then so is G.

2. We shall recall some definitions, results and notations on formal Lie groups in [3, Chap. III]. We denote by H^* the formal Lie group associated with an algebraic subgroup H of $GL(n, K)$. Let f be a rational homomorphism of H into an algebraic linear group H_1. Then there exists a corresponding homomorphism $\boldsymbol{f}$ of H^* into $H_1{}^*$ and $\boldsymbol{f}(H^*)=f(H)^*$. If N is the kernel of f, then N^* is the kernel of $\boldsymbol{f}$. Given an element s of $GL(n, K)$, we denote by $\boldsymbol{a}_s$ the automorphism of $GL^*(n, K)$ corresponding to the inner automorphism of $GL(n, K)$ induced by s. If H is connected, then H^* is solvable (resp. nilpotent, commutative) if and only if H is solvable (resp. nilpotent, commutative). A formal Lie group over K is called representable provided it is isogenous to a subgroup of the formal Lie group $GL^*(n, K)$. The quotient group of a formal Lie group by its center is always representable. For a subgroup G of $GL^*(n, K)$, the algebraic hull $\mathfrak{A}(G)$ is solvable (resp. nilpotent, commutative) if and only if G is solvable (resp. nilpotent, commutative). $\mathfrak{A}(G)^*$ is denoted by $\mathfrak{A}^*(G)$. It is known that $DG=\mathfrak{A}^*(DG)=D(\mathfrak{A}(G))^*$. For a connected algebraic linear group H, we have $\mathfrak{A}(H^*)=H$. The subgroups of any formal Lie group form