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On the Theory of Differentials on Algebraic Varieties

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Introduction

In our previous paper [7](1) we have treated a theory of differentials in commutative rings. In this paper we shall discuss some applications of the foregoing results to problems in algebraic geometry. Let X be a variety and x a point on X and let $\mathcal{O}_x$ be the local ring of x on X. We shall call $D_k(\mathcal{O}_x)=\Omega_x$ the module of local differentials(2) at x, k being the universal domain of our algebraic geometry. In a natural way we can introduce on the set-theoretic union $\Omega=\bigcup_{x\in X}\Omega_x$ a suitable topology in such a way that Ω turns out to be an algebraic coherent sheaf on X. If x is a simple point of X, Ω_x is a free module over $\mathcal{O}_x$ and hence has no torsion. Then we can identify Ω_x with a submodule composed of the differentials of the function field K of X over k. Hence if X is a non-singular variety the sheaf introduced above is identical with the sheaf of germs of regular differentials of degree 1. On the other hand if x is a singular point of x, Ω_x may have torsion in general and some new phenomena take place when we treat the variety with singularities. Although we have no intention to treat the torsion problem here we will present an example to indicate the difference(3). In §2 we deal with the adjoint map associated with a morphism f of a variety Y into X. There we shall introduce two local adjoint maps denoted by f^* and f^{**} respectively. It is one of the purposes of this work to give foundations on the theory of differentials on algebraic varieties based on the theory of local differentials. Hence some known results will be presented with an entirely new proof. The contents of §3 is running along this line, and the existence of invariant differentials on group varieties will be proved within the scope of our method. Though most of the results in this paragraph are not new our formulation is helpful for further discussion. In §4 we shall prove an exact sequence related with the module of local differentials, and it is useful when we discuss the injection of a subvariety Y into the ambiant variety X. Thanks to the exact sequence given in §4 we can prove that the non-existence of non-trivial section of $H^0(X, \mathscr{P}\Omega)$ and $H^0(X, \mathscr{P}/\mathscr{P}^2)$ will be sufficient conditions for ι^* to be a monomorphism, where $\mathscr{P}$ is the sheaf of ideals defined by the subvariety Y and ι is the injec-

(1) The number in the bracket refers to the bibliography at the end of the paper.

(2) For the notion of the module of differentials in commutative rings the readers are expected to refer the article [7].

(3) See the Example in §1.