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Convex Functionals in a Topological Vector Space

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A convex functional on a convex domain of a topological vector space is continuous if it is bounded above in an open subset, and then it becomes locally uniformly continuous [1]. W. Orlicz and Z. Ciesielski have shown [3] that any sequence of convex functionals on a convex domain of a Banach space is equicontinuous if it is bounded at each point of the domain.

In this paper a topological vector space E, locally convex or not, is called a t_0-space if it satisfies the following condition:

(t_0): *Any absorbing convex symmetric closed subset of E is a neighborhood of* 0 *in* E.

Any barrelled space and any topological vector Baire space belong to this type.

In section 1 we shall first prove that if a family of convex, continuous functionals on a convex domain of a t_0-space is bounded above at each point and is bounded at a point, it is equicontinuous. We then extend the theorem of W. Orlicz and Z. Ciesielski to a case of t_0-spaces. In section 2, with the aid of these results, we shall discuss the conditions sufficient for a separately continuous functional defined in a convex domain of a product space to be continuous. They also are extended to a family of functionals.

Throughout this paper a space is understood to be a topological real vector space and any functional is assumed to be real-ralued.

§1. We shall say that a functional f on a convex domain is convex if for any $x, y \in D$ the inequality $f(\lambda x+\mu y)\leq\lambda f(x)+\mu f(y)$ holds, where $\lambda+\mu=1$, $0\leq\lambda, \mu\leq 1$. A functional f is bounded in a set S if there exists a constant C such that $x \in S$ implies $|f(x)|\leq C$. f is locally bounded in a domain if there exists a neighbourhood of each point of the domain on which f is bounded. A family $\{f_\alpha\}_{\alpha\in A}$ of functionals is bounded at a point x if there exists a constant C such that $|f_\alpha(x)|\leq C$ holds for every $\alpha \in A$. It is uniformly bounded in a set S if there exists a constant C such that $x \in S$ implies $|f_\alpha(x)|\leq C$ for every $\alpha \in A$, where C does not depend on x. $\{f_\alpha\}_{\alpha\in A}$ is locally uniformly bounded in a domain if there exists a neighbourhood of each point of the domain in which $\{f_\alpha\}_{\alpha\in A}$ is uniformly bounded. The boundedness above (resp. below) of a functional or a family of functionals may be defined in an obvious manner. $\{f_\alpha\}_{\alpha\in A}$ is equicontinuous at a point x if, for any given $\varepsilon>0$, there exists a neighbourhood $K(x)$ of x such that $x' \in K(x)$ implies $|f_\alpha(x)-f_\alpha(x')|<\varepsilon$ for every $\alpha \in A$,