On Loop Extensions of Groups and M-cohomology Groups

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(Received September 20, 1963)

1. Introduction. The extension problem of groups has been investigated by many authors. Especially S. Eilenberg and S. MacLane studied this problem by using the cohomology theory in [8][1] and [9]. R. H. Bruck studied the problem of the Moufang loop-extensions of an abelian group by a Moufang loop, the central Moufang extensions, by using the factor set in [6].

In this paper, we shall investigate such a loop extension L of a group G by a group Γ as satisfies the following conditions: i) L is a Bol-Moufang loop (i.e. $a[b(ac)]=[a(ba)]c$ in L), ii) L has G as a normal subgroup in its nucleus, iii) $L/G \cong \Gamma$. Let us call the above extension the *BM-extension of G by Γ*. For this purpose of ours, we shall construct a new cohomology group (named *M-cohomology group*). This new *M*-cohomology group will enable us to discuss our extension problem in parallel with the ordinary extension problem of groups, and to make the Bruck's result clearer, when Γ is a group. Moreover, in the same way as that used in the group extension, we shall be able to treat the case where the group G is non-abelian.

In §§2 and 3, we shall study: (i) the necessary and sufficient conditions for the existence of the *BM*-extension of G by Γ; and (ii) the conditions for two *BM*-extensions to be equivalent. Two *BM*-extensions L_1 and L_2 are equivalent if there exists an isomorphism between them, under which G and each coset of L_1/G and L_2/G are invariant. In order to classify the *BM*-extensions, §4 will be devoted to the construction of the *M*-cohomology group H^{*n} which corresponds to the ordinary cohomology group in the group extensions. By using our *M*-cohomology group, we shall classify all *BM*-extensions of an abelian group G by a group Γ in §5. §6 is concerned with the properties of the element of H^{*3} which is determined by the homomorphism $\theta : \Gamma \rightarrow \mathrm{Aut}(G)/\mathrm{In}(G)$ induced by the *BM*-extension. By using these properties, in §7 we shall study the necessary and sufficient condition for the existence of the *BM*-extension of a non-abelian group G by a group Γ, and when such *BM*-extension exists, we shall show that all non-equivalent *BM*-extensions correspond one-to-one to the elements of the second *M*-cohomology group H^{*2} (Γ, C), (C is the center of G). The last §8 will contain some properties of the third *M*-cohomology group.

2. BM-extensions of groups. In this section, we obtain the necessary and sufficient conditions for the existence of the *BM*-extension L of a group

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(1) The number in the brackets refers to the references at the end of this paper.