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A Computation of Extremal Length in an Abstract Space

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In [2] and [3] we computed the extremal length of harmonic subflows in an n-dimensional $\mathscr{E}$-space. In this paper we shall compute the extremal length of a certain class of measures in an abstract space. The main results of [3] are special cases of the results in the present paper.

1. Let X be an abstract space and $\mathfrak{A}$ be a σ-field[1)] of subsets of X. By a measure in this paper we shall mean a non-negative countably additive set-function. Let μ be a measure on $\mathfrak{A}$. With each $x\in X$, we associate an abstract space Y_x, a σ-field $\mathfrak{B}_x$ of subsets of Y_x and a measure ν_x defined on $\mathfrak{B}_x$. We shall denote by Z the set of all couples (x, y), $x\in X$, $y\in Y_x$. Suppose that there is a σ-field $\mathfrak{C}$ of sets in Z[2)] which contains all sets of the form $\{(x, y);\ x\in A\in\mathfrak{A}, y\in Y_x\}$[3)] and which, for every $E\in\mathfrak{C}$, satisfies

(1) $E_x=\{y\in Y_x;\ (x, y)\in E\}$ belongs to $\mathfrak{B}_x$ for every $x\in X$ not belonging to $A_E\in\mathfrak{A}$ with $\mu(A_E)=0$,[4)]

(2) $\nu_x(E_x)$ is an $\mathfrak{A}$-measurable function defined on $X-A_E$. We set

$$\alpha(E)=\int_X \nu_x(E_x)\,d\mu(x) \qquad \text{for } E\in\mathfrak{C}.$$

If $E^{(1)}, E^{(2)}, \cdots$ are mutually disjoint sets of $\mathfrak{C}$, then $\alpha(\bigcup_n E^{(n)})=\sum_n \alpha(E^{(n)})$. Thus, α is a measure on $\mathfrak{C}$. If f is non-negative and $\mathfrak{C}$-measurable, it is inferred that $f(x, y)$ is a $\mathfrak{B}_x$-measurable function of y on Y_x for μ-a.e. $x\in X$, that $\int_{Y_x} f\,d\nu_x$ is an $\mathfrak{A}$-measurable function defined for μ-a.e. $x\in X$ and that

$$\int f\,d\alpha=\int_X\left(\int_{Y_x} f\,d\nu_x\right)d\mu(x).$$

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1) This means that $\mathfrak{A}$ is not empty, $A\in\mathfrak{A}$ implies $X-A$ and $A_1, A_2, \ldots\in\mathfrak{A}$ implies $\bigcup_n A_n\in\mathfrak{A}$. Sometimes, it is called a Borel field or σ-algebra.

2) The existence of $\mathfrak{C}$ will be discussed in Section 6.

3) Any set of this form satisfies conditions (1) and (2) imposed below, because $Z\in\mathfrak{C}$ satisfies condition (2).

4) This fact will be expressed as "for μ-a.e. $x\in X$".