On Decompositions of Riemannian Manifolds.

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Introduction. In a homogeneous space $G/H$, each of its points is a coset of a closed subgroup $H$ of a Lie group $G$, that is, $G/H$ is based on the decomposition of $G$ into the leaves (maximal integral manifolds) of the Lie algebra of $H$. Generalizing, from this point of view, the notion of a homogeneous space, we have that of a foliation $M/\mathcal{M}$ in the sense of R. Palais [4], which consists of leaves of an involutive distribution $\mathcal{M}$ on a differentiable manifold $M$, together with the topology induced from $M$ (see p. 82). Foliations in more general spaces have been treated by C. Ehresmann [2], G. Reeb [5] and A. Haefliger [3].

In the present paper we shall investigate the decomposition of a Riemannian manifold $M$ into the leaves of an involutive distribution $\mathcal{M}$ which has the involutive orthogonal complement $\mathcal{M}^*$. At first it will be tried to represent the foliation $M/\mathcal{M}$ with a leaf $V^*$ of $\mathcal{M}^*$. This requires that the leaf should meet all the leaves of $\mathcal{M}$. We shall find a sufficient condition of that in terms of certain quantities related to a family of geodesic curvatures (Theorem 2). Under this condition, for any simply connected leaf $V^*$ of $\mathcal{M}^*$, we have the relation:

$$M/\mathcal{M} \cong V^*/G(V^*)$$

(Theorem 3), where $G(V^*)$ denotes the group of diffeomorphisms of $V^*$ which make the intersection of $V^*$ and each leaf of $\mathcal{M}$ invariant. Let $H_p$ be the subgroup of $G(V^*)$ consisting of elements which make a point $p$ invariant. Then, if $H_p$ and $H_q$ are conjugate subgroups, it will be shown that the leaves of $\mathcal{M}$ through $p$ and $q$ are diffeomorphic. And, if one of the leaves of $\mathcal{M}$ is simply connected, if the leaves of $\mathcal{M}$ through $p$ and $q$ are homeomorphic, then $H_q$ and $H_p$ are isomorphic (Theorem 4). Finally we shall show that, when $G(V^*)$ is abelian, its elements can be extended to diffeomorphisms of $M$ which make the decomposition of $M$ invariant (Theorem 5).

1. Let $M$ be an $n$-dimensional differentiable\(^1\) manifold with countable base. Let us be given an $m$-dimensional involutive distribution $\mathcal{M}$ on $M$. Since, as is well known, $M$ can be given a Riemannian structure, we have, at any point

\(^1\) By “differentiable” we always mean “of class $C^\infty$".