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Lie Algebras which have Few Derivations

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Introduction

Let L be a Lie algebra over a field of characteristic 0 and let $D(L)$ be the derivation algebra of L. Let $I(L)$ and $C(L)$ be respectively the sets of all inner derivations and of all central derivations of L. In the paper [10], we studied the relationship between the structures of $D(L)$ and L and among other results we showed some results on Lie algebras L which have as few derivations as possible, that is, such that $D(L)=I(L)+C(L)$. It is furthermore natural to make a search for the properties of Lie algebras L such that $D(L)=I(L)^*+C(L)$ where $I(L)^*$ is the algebraic hull of $I(L)$, that is, Lie algebras which have few derivations. The purpose of this paper is to study such a type of Lie algebras.

There actually exists a Lie algebra L such that $D(L)=I(L)^*+C(L)$ but $D(L)\neq I(L)+C(L)$, as will be shown in Section 5. Owing to Lemma 1 in [10] which states that $I(L)^*=\mathrm{ad}_L L^*$ for a linear Lie algebra L, for such Lie algebras we can show the results analogous to those for Lie algebras which have as few derivations as possible.

In [10], generalizing a result of G. Leger [7], we showed that if $D(L)=I(L)+C(L)$ then the radical of L is not quasi-cyclic or is the center of L. We shall give the corresponding results with sharper assertions. Namely, we shall show that, if $D(L)=I(L)^*+C(L)$, then the radical R of L is the direct sum of a central ideal of L and of an ideal R_1 which has no abelian direct summands and all semisimple elements of the radical of $D(R_1)$ are contained in $I(R_1)^*$, and that, if R is further nilpotent, the radical of $D(R_1)$ consists precisely of the nilpotent elements (Theorem 2). It will also be shown that for a Lie algebra L such that $D(L)=I(L)^*$ we have similar statements with $R=R_1$ (Theorem 3). As one of the applications of these results we shall show that any non-abelian nilpotent Lie algebra which is quasi-cyclic or whose dimension is less than 6 cannot be the radical of a Lie algebra L such that $D(L)=I(L)+C(L)$ (Corollary 2 to Theorems 2 and 3).

We shall further prove that $D(L)=I(L)^*+C(L)$ if and only if this is the case for every direct summand of L (Theorem 1) and clarify the structure of Lie algebras whose radicals have few derivations (Theorem 4).