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Some Containment Relations between Classes of Ideals in an Integral Domain

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I. Introduction: Throughout this paper D will denote an integral domain with $1 \neq 0$ and quotient field K. An ideal A of D is called a valuation ideal provided there exists a valuation ring D_v such that $D \subset D_v \subset K$ and $AD_v \cap D = A$ ([10; 340]). Denote by $\mathcal{Q}$ the set of primary ideals of D, by $\mathcal{V}$ the set of valuation ideals of D, by $\mathcal{S}$ the set of semi-primary ideals of D, (i.e. ideals with prime radical) and by $\mathcal{PP}$ the set of prime powers of D. The significance of the various containment relations between these classes of ideals has been investigated in [1], [2], [5], [6] and [8]. The notion of valuation ideal can be generalized by replacing the valuation ring D_v in the above definition by other types of domains, e.g., Prüfer domains, almost Dedekind domains, Dedekind domains, rank one valuation rings, and rank one, discrete valuation rings (which are the generalizations considered in this paper). The purpose of this paper is to investigate certain of the containment relations between the classes of ideals obtained in this manner and the classes listed above. We will usually follow [9] and [10] in matters of notation and definitions. Containment will be denoted by $\subset$ and proper containment by $<$. An ideal A of D is proper provided $(0) < A < D$.

II. Preliminaries. A domain D is called a Prüfer domain provided D_p (the quotient ring of D with respect to the prime ideal P, [9; 228]) is a valuation ring for each proper prime ideal P in D, (see [1], [11; 554], [12; 127], and [13]) and is called an almost Dedekind domain provided D_p is a rank one, discrete valuation ring (i.e., a valuation ring which is a Dedekind domain) for each proper prime ideal P in D (see [3] and [7]). An ideal A of D is a Prüfer ideal if there exists a Prüfer domain J such that $D \subset J \subset K$ and $AJ \cap D = A$. Almost Dedekind ideals, Dedekind ideals, etc… are defined in an analogous manner. Denote by $\mathcal{P}$, $\mathcal{D}$, and $\mathcal{A}$, the set of Prüfer ideals, Dedekind ideals, and almost Dedekind ideals, respectively of the domain D.

In Section III necessary and sufficient conditions are given in order that $\mathcal{P} \subset \mathcal{Q}$, $\mathcal{P} = \mathcal{Q}$, $\mathcal{P} \subset \mathcal{S}$, $\mathcal{P} = \mathcal{S}$, $\mathcal{A} \subset \mathcal{S}$, $\mathcal{A} = \mathcal{S}$, $\mathcal{P} \subset \mathcal{PP}$. Furthermore, it is shown that D is Dedekind if and only if each proper ideal of D is a Dedekind ideal. In Section IV it is shown that D is almost Dedekind if and only if $\mathcal{Q} \subset \mathcal{D}$ and proper prime ideals of D are maximal. Also, the prime ideal structure of D is studied in case that $\mathcal{S} \subset \mathcal{V}$.