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An Application of the Minimax Theorem to the Theory of Capacity

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Let K be a compact Hausdorff space and $\Phi(x, y)$ be an extended real-valued lower semicontinuous function on $K\times K$ which does not assume the value $-\infty$. We denote by $\mathscr{U}_K$ the class of non-negative unit Radon measures on K, and by S_μ the support of a measure μ. The potentials

$$\int\Phi(x, y)\,d\mu(y) \qquad \text{and} \qquad \int\Phi(y, x)\,d\mu(y)$$

will be denoted by $\Phi(x, \mu)$ and $\Phi(\mu, x)$ respectively. Our aim in this paper is to prove

THEOREM. *It holds that*

$$\inf_{\mu\in\mathscr{U}_K}\sup_{x\in S_\mu}\Phi(x, \mu)=\inf_{\mu\in\mathscr{U}_K}\sup_{x\in S_\mu}\Phi(\mu, x) \tag{1}$$

and

$$\sup_{\mu\in\mathscr{U}_K}\inf_{x\in S_\mu}\Phi(x, \mu)=\sup_{\mu\in\mathscr{U}_K}\inf_{x\in S_\mu}\Phi(\mu, x). \tag{2}$$

REMARK. The reciprocal of the value in (1) is taken as the definition of capacity in the Newtonian case, namely, when $\Phi(x, y)=|x-y|^{-1}$ in the Euclidean space E_3. In this case both sides of (2) are always equal to ∞ and hence (2) is trivially true. In case $\Phi(x, y)$ is (finite-valued) continuous on $K\times K$, either one of (1) and (2) follows from the other because

$$\sup_{\mu\in\mathscr{U}_K}\inf_{x\in S_\mu}\begin{matrix}\Phi(x, \mu)\\ \Phi(\mu, x)\end{matrix}=-\inf_{\mu\in\mathscr{U}_K}\sup_{x\in S_\mu}\begin{matrix}(-\Phi(x, \mu))\\ (-\Phi(\mu, x)).\end{matrix}$$

PROOF OF THE THEOREM. We may assume $\Phi>0$ without loss of generality. We shall denote the left and the right hand sides of (1) by α and β respectively. First we consider the case where Φ is finite-valued and K consists of a finite number of points by induction. The case when K consists of one point is trivial. Suppose that (1) is true when K contains exactly n points, and let us consider the case where K consists of $n+1$ points. We can express $\mu\in\mathscr{U}_K$ by