J. Sci. Hiroshima Univ. Ser. A-I
30 (1966), 31-39

A Generalization of Duality Theorem in the Theory of Linear Programming

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(Received March 15, 1966)

We consider an $m\times n$ matrix (a_{ij}), a vector $\boldsymbol{b}$ with components $b_1, \cdots, b_m$ and a vector $\boldsymbol{c}$ with components $c_1, \cdots, c_n$. We denote by $\mathscr{M}$ the set of vectors $\boldsymbol{u}$ having non-negative components $u_1, \cdots, u_n$ and satisfying $\sum_{j=1}^{n} a_{ij}u_j \leqq b_i$ $(i=1, \cdots, m)$, and by $\mathscr{M}'$ the set of vectors $\boldsymbol{v}$ having non-negative components $v_1, \cdots, v_m$ and satisfying $\sum_{i=1}^{m} a_{ij}v_i \geqq c_j$ $(j=1, \cdots, n)$. We set

$$M=\sup_{\boldsymbol{u}\in\mathscr{M}} \boldsymbol{c}\cdot\boldsymbol{u}'=\sup_{\boldsymbol{u}\in\mathscr{M}} \sum_{j=1}^{n} c_j u_j \qquad \text{if } \mathscr{M}\neq\emptyset \text{ (the empty set)}$$

and

$$M'=\inf_{\boldsymbol{v}\in\mathscr{M}'} \boldsymbol{b}\cdot\boldsymbol{v}'=\inf_{\boldsymbol{v}\in\mathscr{M}'} \sum_{i=1}^{m} b_i v_i \qquad \text{if } \mathscr{M}'\neq\emptyset\ .$$

The well-known duality theorem in the theory of linear programming asserts that, if $\mathscr{M}\neq\emptyset$ and $M<\infty$, then $\mathscr{M}'\neq\emptyset$ and $M=M'$.

We shall generalize this theorem in the present paper. Let X and Y be compact Hausdorff spaces and $\Phi(x, y)$ a universally measurable[1)] function on $X\times Y$ which is bounded below. Let $g(x)$ be a universally measurable function on X which is bounded below and $f(y)$ a universally measurable function on Y which is bounded above.

Under these general circumstances let $\mathscr{M}$ be the class of all non-negative Radon measures[2)] μ on Y satisfying

$$\int_Y \Phi(x, y)d\mu(y)\leqq g(x) \qquad \text{on } X.$$

Such a measure is called *feasible*. In case $\mathscr{M}$ is not empty, we set

$$M=\sup_{\mu\in\mathscr{M}}\int f(y)d\mu(y)\ .$$

1) A function in a compact space is universally measurable if it is measurable with respect to all Radon measures.

2) A measure means a non-negative Radon measure in this paper unless otherwise stated.