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A Note on Normal Ideals

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§ 1. Introduction

In [3], p. 85 F. Maeda writes $a \triangledown b$ in a lattice L with 0 to denote the fact that $a \wedge b = 0$ and $(a \vee x) \wedge b = x \wedge b$ for all x in L. He then uses this relation to investigate direct sum decompositions of such lattices. If L is modular the relation $\triangledown$ is symmetric and the mapping $S \rightarrow S^{\triangledown} = \{f: s \triangledown f$ for all $s \in S\}$ induces a Galois connection in the lattice $I(L)$ of all ideals of L. The Galois closed objects (i.e., those ideals S such that $S = S^{\triangledown\triangledown}$) are called *normal ideals*. In a *general continuous geometry* (see [3], p. 90) the normal ideals play a role analogous to that played by the center of a continuous geometry. In this note we investigate normal ideals in a more general setting. In § 2 we show that in a lattice L with 0, an ideal J is in the center of $I(L)$ if and only if it is a direct summand of L. In § 3 we use the fact that the relation $\triangledown$ is symmetric in a relatively complemented lattice with 0 to define normal ideals in such a lattice. We then show that if L is a relatively complemented lattice with 0 and 1, then the center of the completion by cuts $\bar{L}$ of L is precisely the set of normal ideals which are kernels of congruence relations. In the case of a complemented modular lattice, the center of $\bar{L}$ is just the set of normal ideals of L. In § 4 these results are extended to the case of an arbitrary relatively complemented lattice with 0.

§ 2. Direct summands

Let $S_1, S_2, \cdots, S_n$ be subsets of a lattice L with 0. Following the terminology of F. Maeda ([3], p. 85) if

(1°) for any element a of L, $a = a_1 \vee \cdots \vee a_n$ with $a_i \in S_i (i = 1, \cdots, n)$,

(2°) $$i \neq j \quad \text{implies} \quad S_j \subseteq S_i^{\triangledown},$$

we say that L is a *direct sum* of $S_1, \cdots, S_n$ and write $L = S_1 \oplus \cdots \oplus S_n$. The subsets $S_1, \cdots, S_n$ will be called *direct summands* of L. By [3], Lemma 1.3, p. 86 every direct summand is an ideal of L. We proceed to show that the direct summands are precisely the central elements of $I(L)$.

Theorem 1. *Let L be a lattice with 0. An ideal J of L is a central element of $I(L)$ if and only if it is a direct summand of L.*