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On a Capacitability Problem Raised in Connection with the Gauss Variational Problem

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§ 1. Introduction with definitions and problem setting

In a locally compact Hausdorff space, there are many ways to consider a set function for compact sets which is similar to the capacity in the classical sense. Starting from such a set function, we can define an inner quantity and an outer quantity. The problem of capacitability is to discuss when they coincide. A very useful tool is the general theory of capacitability which was estabilished by G. Choquet [2].

In this paper we shall examine the capacitability problem in relation to the Gauss variational problem. More precisely, let Ω be a locally compact Hausdorff space and $\Phi(x, y)$ be a lower semicontinuous function on $\Omega\times\Omega$. Throughout this paper, we shall assume that Φ takes values in $[0, +\infty]$. A measure μ will be always a non-negative Radon measure and $S\mu$ the support of μ. The potential of μ is defined by

$$\Phi(x, \mu)=\int\Phi(x, y)d\mu(y)$$

and the mutual energy of μ and ν is defined by

$$(\nu, \mu)=\int\Phi(x, \mu)d\nu(x).$$

We call (μ, μ) simply the energy of μ. Let $\mathscr{E}$ be the class of all measures with finite energy and put

$$\mathscr{E}_A=\{\mu;\ \mu\in\mathscr{E},\ S\mu \text{ is compact and } S\mu\subset A\}.$$

We note that each measure in $\mathscr{E}_\Omega$ has a compact support and hence $\mathscr{E}_\Omega\neq\mathscr{E}$ in general. The kernel Φ is assumed, unless otherwise stated, to be *of positive type*, i.e. $\Phi(x, y)=\Phi(y, x)$ for all $x, y\in\Omega$ and $(\mu, \mu)+(\nu, \nu)-2(\nu, \mu)\geqq 0$ for all $\mu, \nu\in\mathscr{E}$. The pseudo-metric $\|\mu-\nu\|=[(\mu, \mu)+(\nu, \nu)-2(\nu, \mu)]^{1/2}$ defines *the strong topology* in $\mathscr{E}$.

A class of measures is called *strongly complete* if any strong Cauchy net in the class converges strongly to an element of the class.

We shall recall the quantities which are related to the capacity and were used by M. Ohtsuka [7]. For a nonzero measure μ, put $V(\mu)=\sup\{\Phi(x, \mu);\ x\in S\mu\}$ and for a nonempty set A define $\mathscr{U}_A$ by $\{\mu;\ S\mu$ is compact, $S\mu\subset A$ and