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On Semigroups, Semirings, and Rings of Quotients)*

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1. Introduction.

There are many theorems known about the imbedding of algebraic structures in quotient structures, that is, structures with all the properties of the original ones in which suitable candidates (cancellable elements) become invertible, and such that every element of the larger structure is a quotient of elements of the original structure. The best-known classical theorem of this sort asserts that an integral domain may be imbedded in a field of quotients. The construction of such a field, using equivalence classes of ordered pairs, has been adapted to prove a number of generalizations, such as those of Öre [10] and Asano [1] for rings and Vandiver [14] for semigroups and semirings.

In a negative direction, we have the theorem of Malcev [8] that not every ring without zero divisors can be imbedded in a division ring. On the other hand, if one is willing to give up associativity, such an imbedding can always be accomplished (Neumann [9]). We will confine our attention to associative structures.

In [2], Asano generalized his own work with a different kind of construction of quotients using partial endomorphisms (which he called simply "operators"). This in turn was extensively generalized by Findlay and Lambek [3]. In recent years there have been many papers devoted to the subject of rings of quotients: see, for example, [7, 11] and references listed in these.

For the purposes of ring theory, constructions via partial homomorphisms and related ideas are surely more elegant and efficient than the old-fashioned, but more concrete, constructions via equivalence classes of ordered pairs. For example, the verifications of associativity and distributivity are trivial when one uses mappings as elements of the quotient structure.

However, the student is usually introduced first (perhaps solely) to the more concrete construction. Thus it is of interest to see this construction in perhaps its most general form, where only the essential ideas are present at each step. In the process, it becomes clear that for purposes of extending a multiplicative structure to include quotients, the accompanying additive structure (if any) is of little or no consequence. Hence imbedding theorems

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