J. Sci. Hiroshima Univ. Ser. A-I
31 (1967), 75-88

On C^∞ Maps which admit Transposed Image of every Distribution

Mitsuyuki Itano and Atsuo Jôichi

(Received February 28, 1967)

Introduction

Let Ω be a non-empty open subset of an N-dimensional Euclidean space R^N. For any $\phi \in \mathcal{D}(R^N)$ such that $\phi \geqq 0$ and $\int \phi(x)dx = 1$ we put $\phi_\lambda(x) = \frac{1}{\lambda^N}\phi\left(\frac{x}{\lambda}\right)$, where λ is a positive real number. For given S, $T \in \mathcal{D}'(\Omega)$ we understand the product ST by $\lim_{\lambda \to 0}(S * \phi_\lambda)T$ in $\mathcal{D}'(\Omega)$ if it exists for every $\phi \in \mathcal{D}(\Omega)$ (Notation $S \bigcirc T$ was used in [5]). The multiplication thus defined is invariant under diffeomorphism.

A distribution S in R^n may be considered as the distribution $S \otimes 1_y$ in $R^n_x \times R^m_y$. Similarly $T \in \mathcal{D}'(R^m)$ is identified with $1_x \otimes T$ in $R^n_x \times R^m_y$. The product $(S \otimes 1_y)(1_x \otimes T)$ exists and is equal to $S \otimes T$ (Proposition 3). If χ is any diffeomorphism of $R^n \times R^m$ onto itself, then $(S \otimes 1_y)(\chi)(1_x \otimes T)(\chi)$ exists and is equal to $\big((S \otimes 1_y)(1_x \otimes T)\big)(\chi)$ [5, p. 163]. Now a question arises as to its converse: if ξ and η are C^∞ maps of $R^n \times R^m$ into R^n and R^m respectively, what is a necessary and sufficient condition in order that the product $S(\xi)T(\eta)$ may be defined for every $S \in \mathcal{D}'(R^n)$ and $T \in \mathcal{D}'(R^m)$? To answer this question, first we have to clear up the meaning of the notation such as $S(\xi)$. To do so, we shall introduce the concept of an admissible map (see Definitions 1, 2 below), which allows us to make an extension of a notion of "a function of functions". Roughly speaking, a C^∞ map χ of an open set Ω into another Ω' is admissible whenever the transposed image $\xi^* S$ of every $S \in \mathcal{D}'(\Omega')$ exists. The question is then answered: If ξ and η are admissible, a necessary and sufficient condition is that the map (ξ, η) of $R^n \times R^m$ into itself has no critical point (Corollary to Theorem 2). From a different approach, though mainly designed for practical purpose, I. M. Gel'fand and G. E. Shilov have developed the detailed discussions on "a distribution of functions" [2, Chapter III].

It is probable that a C^∞ map is admissible if and only if it has no critical point. We did not succeed in deciding whether it is true or not. In certain special instances we can show that the conjecture holds true (Example 2, Proposition 8).

Section 1 is mainly devoted to the preliminary discussions on transposed images and admissible maps. Owing to these notions we can study the pro-