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On Certain Properties of Lie Algebras

Atsuo Jôichi

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Introduction. I. M. Singer [1] has introduced the following condition for a Lie algebra L: (A) Any pair of elements x, y of L such that $[x, [x, y]]=0$ satisfies $[x, y]=0$. M. Sugiura [2] called a Lie algebra satisfying this condition to be an (A)-algebra and proved, among other results, that a Lie algebra L over a field of characteristic 0 is an (A)-algebra if and only if any $x \in L$ such that $(\mathrm{ad}\, x)^k=0$ for some $k \geqq 2$ satisfies $\mathrm{ad}\, x=0$. On the other hand, S. Tôgô [3] has considered a Lie algebra L satisfying the condition that $(\mathrm{ad}\, x)^2=0$ implies $\mathrm{ad}\, x=0$, and has given an example of such a Lie algebra which is solvable but not abelian. This is not an (A)-algebra since any solvable (A)-algebra is abelian ([1], [2]). Thus we are led to consider a Lie algebra which satisfies the condition that $(\mathrm{ad}\, x)^k=0$ implies $\mathrm{ad}\, x=0$ for a fixed integer $k \geqq 2$. We shall call such a Lie algebra to be an (A_k)-algebra. In this paper we shall investigete the properties of (A_k)-algebras.

It will be shown that a solvable (resp. nilpotent) Lie algebra over a field of characteristic 0 is an (A_k)-algebra with $k \geqq 3$ (resp. $k \geqq 2$) if and only if it is abelian. We shall show that an (A_2)-algebra is not always an (A_k)-algebra with $k \geqq 3$, much less an (A)-algebra. As to (A_k)-algebras with $k \geqq 3$, if the basic field is algebraically closed and of characteristic 0, we can show that an (A_k)-algebra is abelian and so an (A)-algebra. A detailed discussion about (A_2)-algebras is also given.

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Notations. We denote by Φ a field of arbitrary characteristic unless otherwise stated and by L a finite dimensional Lie algebra over a field Φ. We denote by $Z(L)$ the center of L and by

$$Z_0(L)=Z(L) \subset Z_1(L) \subset Z_2(L) \subset \cdots \subset Z_n(L) \subset \cdots$$

the ascending central series of L. For a subspace U of L the centralizer of U in L will be denoted by $C(U)$.

1. We start with the following

Definition 1. *For an integer $k \geqq 2$, we call a Lie algebra L over a field Φ*