J. Sci. Hiroshima Univ. Ser. A-I
**31** (1967), 17-23

# *Dimensions of the Derivation Algebras of Lie Algebras*

Shigeaki Tôgô

(Received February 27, 1967)

Let $L$ be a Lie algebra over a field of arbitrary characteristic. In the paper [3], J. Dozias has shown that if $L \neq [L, L]$ then the dimension of the derivation algebra $\mathfrak{D}(L)$ of $L$ is not less than dim $L$. In this paper, making use of the method of constructing outer derivations of $L$ which has been shown in [6], we shall give some effective estimates of $\dim \mathfrak{D}(L)$.

In Section 1 we shall recall some results which have been already shown in [4] and [6]. In Section 2 we shall give several estimates of $\dim \mathfrak{D}(L)$. If $Z(L)$ is the center of $L$ and $C([L, L])$ is the centralizer of $[L, L]$ in $L$, then one of the estimates is that if $L \neq [L, L]$

$$\dim \mathfrak{D}(L) \geq \dim L + \max \{\dim Z(L) \dim L/[L, L] - \dim C([L, L]), 0\}.$$

In Section 3 we shall give several examples which are connected with the results in Section 2.

**1.** Throughout the paper we denote by $\Phi$ a field of arbitrary characteristic unless otherwise stated and by $L$ a finite dimensional Lie algebra over a field $\Phi$. We denote by $\mathfrak{D}(L)$ the derivation algebra of $L$, that is, the Lie algebra of all the derivations of $L$ and by $\mathfrak{J}(L)$ the ideal of $\mathfrak{D}(L)$ consisting of all the inner derivations of $L$. We denote by $Z(L)$ the center of $L$ and, for a subalgebra $H$ of $L$, by $C(H)$ the centralizer of $H$ in $L$. As usual $[L, L]$ will be denoted by $L^2$.

In the next section we need the following two results.

Lemma 1. *Let $L$ be a Lie algebra over a field $\Phi$ and let $M$ be an ideal of $L$ of codimension* 1 *containing $Z(L)$. Then*:

(i) $[L, Z(M)] \subset Z(M)$ *and*

$$\dim Z(M) = \dim Z(L) + \dim [L, Z(M)].$$

(ii) *If $L = (e) + M$ and $Z(L) \neq (0)$, every endomorphism of $L$ sending $e$ to any element of $Z(M) \backslash [L, Z(M)]$ and $M$ into* (0) *is an outer derivation of $L$.*

Lemma 2. *Let $L$ be a Lie algebra over a field $\Phi$. If $L$ is the direct sum of the ideals $L_1$* and *$L_2$, then*

$$\dim \mathfrak{D}(L) = \dim \mathfrak{D}(L_1) + \dim \mathfrak{D}(L_2) + \dim Z(L_1) \dim L_2/L_2^2 + \\ + \dim Z(L_2) \dim L_1/L_1^2.$$