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Idempotent Ideals and Unions of Nets of Prüfer Domains

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0. Introduction

In this paper, all rings considered are assumed to be commutative rings with an identity element. It is known that an integral domain D may contain an idempotent proper ideal A. But when this occurs, A is not finitely generated [21, p. 215], so that D is not Noetherian. Also, it is easy to show that for any positive integer k there exists a ring R which is not a domain and such that R contains an ideal A with the property that $A \supset A^2 \supset \cdots \supset A^k = A^{k+1} = \cdots$. Whether an integral domain R with this property exists is a heretofore open question which we answer affirmatively in §2.

Nakano in [16] has considered the problem of determining when an ideal of D is idempotent, where D is the integral closure of Z, the domain of ordinary integers, in an infinite algebraic number field. In fact, the paper [16] is one of a series of papers which Nakano has written concerning the ideal structure of D. In [18], Ohm has generalized and simplified many of Nakano's results from [16] and [17], showing that as far as the structure of the set of primary ideals of D is concerned, the assumption that D is the integral closure of Z in an algebraic number field is superfluous; the essential requirement on D being that it is a *Prüfer domain* according to the following definition: The integral domain J is a Prüfer domain if for each proper prime ideal P of J, J_P is a valuation ring; equivalently, J is a Prüfer domain if each nonzero finitely generated ideal of J is invertible [10, p. 554].

Following Ohm's example, we show in §3 that most of Nakano's results in [16] carry over to the case when D is the integral closure of a fixed Prüfer domain D_0 in an algebraic extension of the quotient field of D_0.

If J is an integral domain with quotient field K, a domain J_0 between J and K will be called an *overring* of J. In case J_0 is a valuation ring, we call J_0 a *valuation overring* of J. We say that J is an *almost Dedekind* domain if for each maximal ideal M of J, J_M is a rank one discrete valuation ring [5], [1].

1. Preliminary results on Prüfer domains.

We list in this section some results in the theory of Prüfer domains