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Overrings of Almost Multiplication Rings

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1. Introduction

In [3] Gilmer introduced the notion of almost Dedekind domain. In [1] Butts and Phillips studied a class of rings with the property that the integral domains in this class are the almost Dedekind domains; they called these rings almost multiplication rings. In this paper we shall obtain several results concerning almost multiplication rings which reduce to known results when stated for almost Dedekind domains. In particular, results are obtained concerning overrings of a certain type of almost multiplication ring.

All rings which we consider in this paper will be commutative rings with unity. Let R be such a ring. We shall always denote by K the total quotient ring of R. If S is a multiplicatively closed set in R, and if S contains no zero-divisors, we shall assume that the ring of quotients R_S is imbedded in K in the natural way. If P is a prime ideal of R we denote by R_P the ring of quotients of R with respect to P, by $S(P)$ the multiplicatively closed set of non-zero-divisors in R but not in P, and by $N(P)$ the set of all $x \in R$ such that $sx=0$ for some $s \in R$ with $s \notin P$. By an *overring* of R we mean a ring T such that $R \subseteq T \subseteq K$. We shall write $A \subset B$ when A is a proper subset of B; an ideal A of R will be referred to as proper if $A \subset R$.

A ring R is called an *almost multiplication ring* (AM-ring) if for every proper prime ideal P of R, each ideal of R_P can be written as a product of prime ideals. If this is the case, then for every proper prime ideal P of R, the ring R_P is either a discrete rank one valuation ring or a special primary ring. Several sets of equivalent conditions for a ring to be an AM-ring have been given by Butts and Phillips [1; thm. 2.0].

An element of a ring R will be called *regular* if it is not a zero-divisor. An ideal A will be called *regular* if it contains at least one regular element; otherwise A will be called a *Z-ideal*. A ring R is said to have *few zero-divisors* if there are only a finite number of maximal elements in the set of Z-ideals of R. This notion was introduced by Davis in [2]. We shall use several times the following consequence of the fact that a ring has few zero-divisors: if R has few zero-divisors then every regular ideal of R is generated by its regular elements, and every finitely generated regular ideal of R has a finite set of generators consisting of regular elements [2; lemma B]. A ring R is called a *quasi-valuation ring* if R has few zero-divisors and if the set of regular