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On a Class of Lie Algebras

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Introduction

In the previous paper [4], we have given an estimate for the dimensionality of the derivation algebra of a Lie algebra L satisfying the condition that $(\mathrm{ad}\, x)^2=0$ for $x \in L$ implies $\mathrm{ad}\, x=0$. Such a Lie algebra will be referred to as an (A_2)-algebra in this paper according to the definition given in Jôichi [2], which investigates the (A_k)-algebras, $k \geqq 2$, with intention to obtain the analogues to the (A)-algebras. He showed that the (A_2)-algebras have a different situation from the other classes of (A_k)-algebras, $k \geqq 3$. But the problem of characterizing the (A_2)-algebras remains unsolved. The purpose of this paper is to make a detailed study of this class of Lie algebras.

It is known [3] that every semisimple Lie algebra over the field of complex numbers contains no non-zero element x with $(\mathrm{ad}\, x)^2=0$. We shall show that every Lie algebra over a field Φ of characteristic $\neq 2$ whose Killing form is non-degenerate has the same property. By making use of this result we shall show that, when the basic field Φ is of characteristic 0, L is an (A_2)-algebra if and only if every element x of the nil radical N such that $(\mathrm{ad}\, x)^2=0$ belongs to the center $Z(L)$, and if and only if L is either reductive, or $L \supset N \supset Z(N)=Z(L) \supseteqq N^2 \neq (0)$ and $(\mathrm{ad}\, x)^2 \neq 0$ for any $x \in N \setminus Z(L)$. This characterization will be used in classifying certain types of solvable (A_2)-algebras. A solvable (A_2)-algebra is not generally abelian. We shall show that if Φ is an algebraically closed field of characteristic 0, then every solvable (A_2)-algebra over a field Φ is abelian. The latter half of the paper will be devoted to the study of solvable (A_2)-algebras, in particular, to the study of solvable (A_2)-algebras L such that $\dim N/Z(L)$ is 2 or 3 and of solvable (A_2)-algebras of low dimensionalities.

§1.

Throughout this paper we denote by L a finite dimensional Lie algebra over a field Φ and denote by R, N and $Z(L)$ the radical, the nil radical and the center of L respectively.

Following the terminology employed in [2], we call L to be an (A_2)-*algebra* provided that it satisfies the following condition:

(A_2) *Every element x of L such that $(\mathrm{ad}\, x)^2=0$ satisfies $\mathrm{ad}\, x=0$, that is, belongs to $Z(L)$.*

We first quote a result shown in Theorem 1 in [2] as the following