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Commutative Rings for Which Each Proper Homomorphic Image is a Multiplication Ring[1)]

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In this paper, all rings considered are assumed to be commutative rings. A ring R is called an *AM-ring* (for allgemeine Multiplikationring) if whenever A and B are ideals of R with A properly contained in B, then there is an ideal C of R such that $A=BC$. An AM-ring R in which $RA=A$ for each ideal A of R is called a *multiplication ring*[2)]. This paper considers a ring R *satisfying property* (*Hm*): Each proper homomorphic image of R is a multiplication ring. Numerous ring-theoretic properties (for example, Noetherian, or proper prime ideals are maximal) are inherited by a ring R if these properties hold in each proper homomorphic image of R. In Section 3 of this paper we show, however, that a ring satisfying (*Hm*) need not be a multiplication ring, and we give a characterization of rings with identity satisfying property (*Hm*). An outline is given for constructing examples of rings with identity satisfying (*Hm*) that are not multiplication rings.

Let R be a ring. We say that R *satisfies property* (*) if each ideal of R with prime radical is primary. Property (*) is considered by Gilmer in [3] and [4] and by Gilmer and Mott in [5]. Closely related to (*) is the *property* (**) which is also studied in [5] and in [1] by Butts and Phillips: *Each ideal of R with prime radical is a prime power.* If every proper homomorphic image of R satisfies property (*) (satisfies property (**)), we say that R *satisfies property* (H*) (*satisfies property* (H**)). In [5] it is shown that an AM-ring satisfies (*) and (**) and that if S is a u-ring, S satisfies (**) if and only if S satisfies (*) and primary ideals of S are prime powers. It follows that if R contains an identity, then R a multiplication ring implies that R satisfies (**) and R satisfying (**) implies that R satisfies (*). Hence, in a ring with identity, (*Hm*) implies (H**) and (H**) implies (H*). For this reason, we consider rings satisfying (H*) in Section 1 and rings satisfying (H**) in Section 2. In particular, rings with identity satisfying (H**) are characterized in Section 2.

The notation and terminology is that of [9] with two exceptions: $\subseteq$ denotes containment and $\subset$ denotes proper containment, and we do not assume that a Noetherian ring contains an identity. If A is an ideal of a ring

1) This paper is a portion of the author's doctoral dissertation, written under the direction of Professor Robert W. Gilmer, Jr. of The Florida State University. This part of the dissertation was written under the direction of Professor Joe L. Mott, while Professor Gilmer was on leave of absence.

2) For a historical development of the theory of multiplication rings see [5, p. 40].