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Note on Outer Derivations of Lie Algebras

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Introduction

Let $\mathfrak{D}$ be the set of Lie algebras L over a field Φ satisfying the conditions that $L \neq L^2$ and $Z(L) \neq (0)$, where $Z(L)$ denotes the center of L. Clearly every non-trivial nilpotent Lie algebra belongs to $\mathfrak{D}$. It is known ([4], [6], [8], [13]) that every $L \in \mathfrak{D}$ has an outer derivation. In [13] we have introduced the notion of Lie algebras of type (T) and shown that every Lie algebra L of type (T) such that $L^{(1)} \neq L^{(2)}$ admits an outer derivation belonging to $\mathfrak{R}$, the radical of the derivation algebra $\mathfrak{D}(L)$. It has been also shown that if $L \in \mathfrak{D}$ is not of type (T) there exists an abelian ideal of $\mathfrak{D}(L)$ containing an outer derivation. From these observations it seems to be interesting to study the case where L is of type (T) such that $L^{(1)} = L^{(2)}$. The main purpose of this note is to give a detailed consideration to the case just mentioned. Some additional remarks will be also given.

In Section 2 we shall show that a Lie algebra L of type (T) such that dim $Z(L) \neq 1$ or Φ is of characteristic 2 admits an outer derivation in $\mathfrak{R}$ and that a Lie algebra L of type (T) such that $L^{(1)} = L^{(2)}$, dim $Z(L) = 1$ and Φ is of characteristic $\neq 2$ admits an outer derivation in $\mathfrak{R}$ if and only if $L^{(1)}$ does (Theorem 2.2). In Section 3, we shall show that a Lie algebra L over a field of characteristic 0 admits a semisimple outer derivation in $\mathfrak{R}$ if the radical of L does (Proposition 3.1), and based on this result, for a Lie algebra L of type (T) such that $L^{(1)} = L^{(2)}$ and dim $Z(L) = 1$, we shall give several properties of the radical of $L^{(1)}$, each of which ensures the existence of a semisimple outer derivation in $\mathfrak{R}$ (Theorem 3.6).

In [12] we have studied the existence of the automorphisms of L, when Φ is of characteristic 0, outside the connected algebraic group such that the corresponding Lie algebra is the algebraic hull $\mathfrak{J}(L)^*$ of $\mathfrak{J}(L)$, the ideal of $\mathfrak{D}(L)$ consisting of all inner derivations of L. The final section 4 will be devoted to the discussions about the existence of the derivations of $L \in \mathfrak{D}$ which are contained in $\mathfrak{R}$ but not in $\mathfrak{J}(L)^*$.

§ 1. Preliminaries and notations

Throughout this note we shall consider a finite dimensional Lie algebra L over a field Φ. We denote by R the radical of L and by $Z(L)$ the center of L.

As in [13], we shall denote by $\mathfrak{D}$ the class of all the Lie algebras L over