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On the Structure of Bialgebras Attached to Group Varieties

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As compared with separable isogenies of group varieties, inseparable ones have a peculiar aspect. Let G and G' be two group varieties and let α be an isogeny of G onto G'. Then the tangential mapping α_* of the tangent space at the unit point of G to that of G' associated with α is not an isomorphism if α is inseparable, whereas α_* is an isomorphism if α is separable. We may say, in the scheme theoretic languages, that an inseparable isogeny α of G has the kernel of α which is a group subscheme of G with the non-reduced structure sheaf. In particular the kernel of a purely inseparable isogeny of G is a group subscheme of G with one point e such that the stalk of the structure sheaf at e is an artinian local ring.

As to the purely inseparable isogenies of height 1, it is known that the kernels of the tangential mappings determine these isogenies. Precisely, let $\mathfrak{g}$ be the Lie algebra of G consisting of the left invariant derivations of G. Then p-subalgebras of Lie of $\mathfrak{g}$ stable under the adjoint representation of G correspond to purely inseparable isogenies of G of height 1. This was obtained essentially by I. Barsotti in [1], and some authors generalized his results (cf. [3], [5] and [11]). Barsotti considered in [1] also kernels of general purely inseparable isogenies of group varieties and used invariant semi-derivations (or hyperderivations in his terminologies) on G as tools, which were introduced by J. Dieudonné for formal Lie groups of a positive characteristic in [6]. However he did not pursue complete results in general cases, and P. Cartier developed some theories on this subject (cf. [2], [3] and [4]).

The aim of this paper is to give a theory of purely inseparable isogenies of group varieties essentially from Barsotti's point of view originated in the paper [1]. The main results are as follows. Let G be a group variety defined over an algebraically closed field k, and denote by $\mathfrak{g}(G)$ the set of left invariant semi-derivations on G. Then we shall show that $\mathfrak{H}(G)=k\oplus\mathfrak{g}(G)$ is a bialgebra over k using the main results in [3], and that the set of isomorphism classes of purely inseparable isogenies of G corresponds bijectively to the set of subbialgebras of $\mathfrak{H}(G)$ of finite dimensions which are stable under the adjoint representation of G. Moreover if $N(\alpha)$ is the corresponding subbialgebra of a purely inseparable isogeny α of G, it will be shown that the affine algebraic group scheme Spec $(N(\alpha)^D)$ is the kernel of α in the scheme theoretic sense, where $N(\alpha)^D$ is the linear dual of the bialgebra $N(\alpha)$.