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Harmonic and Full-harmonic Structures on a Differentiable Manifold

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Introduction

Let Ω be a bounded domain in the d-dimensional euclidean space ($d \geqq 2$). G. Stampacchia [17] (also, C. B. Morrey Jr. [14] and O. A. Ladyzhenskaya and N. N. Ural'tzeva [9]) discussed properties of solutions of a second order elliptic partial differential equation on Ω of the form

(1) $$Lu \equiv -\sum_{i,j} \frac{\partial}{\partial x_j}\left(g_{ij}\frac{\partial u}{\partial x_i} + b_j u \right) + \sum_i a_i \frac{\partial u}{\partial x_i} + qu = 0$$

with not necessarily continuous coefficients. In fact, Stampacchia only assumed that coefficients g_{ij}, a_i, b_j and q are measurable functions on Ω satisfying the following conditions (2) and (3):

(2) $$\sum g_{ij}\xi_i\xi_j \geqq \nu|\xi|^2 \quad \text{for some } \nu > 0 \quad \text{and} \quad |g_{ij}| \leqq M.$$

(3) $a_i \in \boldsymbol{L}^d(\Omega)$, $b_j \in \boldsymbol{L}^r(\Omega)$, $q \in \boldsymbol{L}^{r/2}(\Omega)$ for $r > d$. (Cf. [9] and [14], in which it is assumed that $a_i \in \boldsymbol{L}^r(\Omega)$. In case $d=2$, this assumption may be necessary; the paper [17] primarily concerns the case $d \geqq 3$.)

On the ground of Stampacchia's work, R.-M. and M. Hervé [7] developed a theory of superharmonic functions associated with the equation (1), under an additional condition:

(4) $$q - \sum_j \frac{\partial b_j}{\partial x_j} \geqq 0 \quad \text{and} \quad q - \sum_i \frac{\partial a_i}{\partial x_i} \geqq 0 \quad \text{in the distribution sense.}$$

In fact, they showed that the continuous solutions of (1) form a harmonic space on Ω in the sense of M. Brelot [1] and then constructed the corresponding Green function on Ω.

In this paper, we take a connected C^1-manifold Ω and consider a contravariant tensor (g^{ij}), contravariant vectors (a^i) and (b^j) and a function q on Ω which locally satisfy conditions (2) and (3). Our differential equation may be written as

(1′) $$Lu \equiv \Delta u - \sum_i a^i \frac{\partial u}{\partial x_i} + \frac{1}{\sqrt{G}} \sum_j \frac{\partial}{\partial x_j}(\sqrt{G}\, b_j u) - qu = 0$$