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Notes on Hausdorff Dimensions of Cartesian Product Sets

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§1. In the present paper we shall be concerned with evaluation of upper and lower bounds of fractional dimensions of Cartesian product sets by means of the fractional dimensions of their components. Various results on this problem have been obtained by A. S. Besicovitch, P. A. P. Moran, J. M. Marstrand and M. Ohtsuka ([1], [4], [6]). One of results is the following: for given α, $0\leqq\alpha\leqq n$, and β, $0\leqq\beta\leqq m$, if E_1 is a subset of R^n with dim $(E_1)=\alpha$ and E_2 is a subset of R^m with dim $(E_2)=\beta$, then $\alpha+\beta\leqq$ dim $(E_1\times E_2)\leqq\min\{n+\beta, m+\alpha\}$.

We use the notation $\Lambda_\alpha(E)$ for the α-dimensional measure of a set E in a Euclidean space. Note that dim $(E)=\alpha$ is equivalent to $\Lambda_{\alpha-\varepsilon}(E)=\infty$ and $\Lambda_{\alpha+\varepsilon}(E)=0$ for any $\varepsilon>0$.

In this note we shall show that for any given α, $0\leqq\alpha<n$, β, $0\leqq\beta<m$, and γ such that $\alpha+\beta<\gamma<\min\{n+\beta, m+\alpha\}$, there exist $E_1\subset R^n$ and $E_2\subset R^m$ which satisfy $0<\Lambda_\alpha(E_1)<\infty$, $0<\Lambda_\beta(E_2)<\infty$ and $0<\Lambda_\gamma(E_1\times E_2)<\infty$.

§2. Let R^n be the n-dimensional Euclidean space and let $h(r)$ be a continuous increasing function of r such that $h(r)>0$ for $r>0$ and $h(\mathrm{o})=0$. The Hausdorff Λ_h-measure of a subset E of R^n is defined as follows. First, for $\rho>0$, we set

$$\Lambda_h{}^{(\rho)}(E)=\inf\{\sum_{\nu=1}^{\infty}h(d_\nu)\},$$

where the infimum is taken over all coverings of E by at most a countable number of n-dimensional open (or closed) cubes I_ν with the sides $d_\nu\leqq\rho$. Then the Hausdorff measure of E is defined by

$$\Lambda_h(E)=\lim_{\rho\to 0}\Lambda_h{}^{(\rho)}(E).$$

If $h(r)=r^\alpha(\alpha>0)$ (resp. $h(r)=1/\log(1/r)$), then we use the notation Λ_α (resp. Λ_o) instead of Λ_h. It is called the α-dimensional measure (resp. logarithmic measure).

The fractional dimension dim (E) of a subset $E\subset R^n$ is defined by

$$\dim(E)=\inf\{\alpha;\ \Lambda_\alpha(E)=0\}.$$