Hiroshima Math. J.
2 (1972), 369–396

A Study of $\mathscr{D}'_{L^2}$-Valued Distributions on a Semi-Axis in connection with the Cauchy Problem for a Pseudo-Differential System

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(Received September 20, 1972)

In a previous paper [10] one of the present authors has investigated the fine Cauchy problem for a system of linear partial differential operators and obtained the following result: Let $\vec{P}(t, x, D_x)$ be an $N\times N$ matrix of linear partial differential operators with coefficients $\in C^\infty(R_{n+1})$. The fine Cauchy problem consists in finding a solution $\vec{u}=(u_1, u_2,\cdots, u_N)$, $u_j\in\mathscr{D}'(R^+_{n+1})$ to the equation

$$D_t\vec{u}+\vec{P}(t, x, D_x)\vec{u}=\vec{f} \qquad \text{in } R^+_{n+1}$$

with initial condition

$$\lim_{t\downarrow 0}\vec{u}(t, x)=\vec{\alpha},$$

when $\vec{\alpha}=(\alpha_1, \alpha_2,\cdots, \alpha_N)$, $\alpha_j\in\mathscr{D}'(R_n)$ and $\vec{f}=(f_1, f_2,\cdots, f_N)$, $f_j\in\mathscr{D}'(R^+_{n+1})$ are arbitrarily given, where $\lim_{t\downarrow 0}\vec{u}$ denotes the distributional boundary value of $\vec{u}$. If there exists a solution $\vec{u}$ for the problem, then $\vec{f}$ must have the canonical extension $\vec{f}_\sim$ over $t=0$ and $\vec{v}=\vec{u}_\sim$ satisfies the equation

$$D_t\vec{v}+\vec{P}(t, x, D_x)\vec{v}=\vec{f}-i\delta\otimes\vec{\alpha}.$$

Conversely, if $\vec{v}=(v_1, v_2,\cdots, v_N)$, $v_j\in\mathscr{D}'_+(R_{n+1})$ is a solution of this equation, then the restriction $\vec{u}=\vec{v}\,|\,R^+_{n+1}$ is a solution for our original Cauchy problem and $\vec{u}_\sim=\vec{v}$. If we replace $\vec{P}(t, x, D_x)$ by $\vec{A}(t)$, an $N\times N$ matrix of pseudo-differential operators [cf. p. 384 for definition], we shall have a right reason to consider the spaces $\mathscr{D}'(R^+_t)((\mathscr{D}'_{L^2})_x)$ and $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$ instead of $\mathscr{D}'(R^+_{n+1})$ and $\mathscr{D}'(R_{n+1})$ respectively. As a result, it will be natural to introduce the boundary value and the canonical extension in a suitable sense.

The present paper is also designed to be the introductory part of our subsequent paper [12] which will appear in this journal.

In Section 1 we discuss the space $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$ and the spaces related to it. These spaces are all reflexive, ultrabornological and Souslin. Section 2 is devoted to discussions concerning the $\mathscr{D}'_{L^2}$-boundary value and the $\mathscr{D}'_{L^2}$-canonical extension. Various alternatives of these notions will also be considered. In Section 3 we shall introduce the operator $\vec{A}(t)$ referred to above and in-