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Immersions and Embeddings of Lens Spaces

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§1. Introduction

Throughout this note q will denote an odd integer >1, and n a positive integer. Let $L^n(q)$ be the $(2n+1)$-dimensional standard lens space mod q, and let $L^\infty(q)$ be $\bigcup_n L^n(q)$, which is the Eilenberg-MacLane space $K(Z_q, 1)$, where Z_q is a cyclic group of order q. Denote by $\iota(\in H^1(K(Z_q, 1); Z_q)\cong Z_q)$ the fundamental class of $K(Z_q, 1)$. The element $y_n(\in H^1(L^n(q); Z_q)\cong Z_q)$ is called the distinguished generator if $y_n=i^*\iota$, where $i^*: H^1(K(Z_q, 1); Z_q)\rightarrow H^1(L^n(q); Z_q)$ is the isomorphism induced by the natural inclusion $i: L^n(q)\rightarrow L^\infty(q)=K(Z_q, 1)$.

For a given $d\in Z_q$, a continuous map $f: L^n(q)\rightarrow L^m(q)$ is said to have degree $d(=\deg(f))$, if $f^*y_m=dy_n$, where $f^*: H^1(L^m(q); Z_q)\rightarrow H^1(L^n(q); Z_q)$ is the homomorphism induced by f, and where y_n and y_m are the distinguished generators. If $n<m$, the set of homotopy classes of maps of $L^n(q)$ in $L^m(q)$ is in one-to-one correspondence with $H^1(L^n(q); Z_q)(\cong Z_q)$. Thus the homotopy class of a map $f: L^n(q)\rightarrow L^m(q)$, $n<m$, is completely characterized by $\deg(f)$.

The first purpose of this paper is to consider the question: "Which homotopy classes of continuous maps $L^n(q)\rightarrow L^m(q)$ contain a differentiable immersion (or a differentiable embedding)?"

S. Feder has investigated in [2] the question on complex projective spaces and H. Suzuki has studied in [10] and [11] the question in the case of higher order non-singular immersions for projective spaces. The problem for general manifolds is treated by E. Thomas [13], [14] and M. Adachi [1].

By the work of M.W. Hirsch [3] and D. Sjerve [8] we see that any map $L^n(q)\rightarrow L^m(q)$ is homotopic to an immersion for $m\geqq n+[n/2]+1$ if q is odd (cf. §2). For $m\leqq n+[n/2]$, we have the following results.

THEOREM A. *Let q be an odd integer >1, and let k be an integer with $0<k\leqq[n/2]$. If a map $f: L^n(q)\rightarrow L^{n+k}(q)$ with degree d is homotopic to an immersion, then*

$$\sum_{i+j=k}(-1)^i\binom{n+i}{i}\binom{n+k+1}{j}d^{2j}$$

is a quadratic residue mod q.