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Note on the Enumeration of Embeddings of Real Projective Spaces

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§ 1. Introduction

Recently, Y. Nomura [12] has studied the enumeration problem of liftings of a given map to a fibration and its application to the enumeration problem of immersions of certain manifolds. In this note, using his results we enumerate the non-zero cross sections of certain vector bundles, and then study the embedding problem of the real projective spaces in the euclidean spaces.

Let ξ be an orientable n-plane bundle over a CW-complex X of dimension less than $n+2$, and let $w_2(\xi)$ be the second Stiefel-Whitney class of ξ. Consider the homomorphisms

$$(1.1)\qquad \begin{aligned} &\Theta_\xi^i\colon H^{i-1}(X; Z)\longrightarrow H^{i+1}(X; Z_2),\\ &\Gamma_\xi^i\colon H^i(X; Z_2)\longrightarrow H^{i+2}(X; Z_2), \end{aligned}$$

of the cohomology groups, defined by

$$\Theta_\xi^i(a)=Sq^2\rho_2 a+\rho_2 a\cdot w_2(\xi),$$

$$\Gamma_\xi^i(b)=Sq^2 b+b\cdot w_2(\xi),$$

where ρ_2 is the mod 2 reduction. Then we prove the following theorem in §§ 2–4, using Nomura's theorem [12, § 2] and the Postnikov factorization of the universal orientable $(n-1)$-sphere bundle $BSO(n-1)\to BSO(n)$.

THEOREM A. *Let $n\geq 6$ and let ξ be an orientable n-plane bundle over a CW-complex X of dimension less than $n+2$ with a non-zero cross section. Then, the set cross (ξ) of (free) homotopy classes of non-zero cross sections of ξ is given by*

$$cross(\xi)=\begin{cases} \operatorname{Ker}\Theta_\xi^n\times\operatorname{Coker}\Theta_\xi^{n-1}, & \textit{if } \Gamma_\xi^{n-1} \textit{ is epimorphic},\\ \operatorname{Ker}\Theta_\xi^n\times\operatorname{Coker}\Theta_\xi^{n-1}\times\operatorname{Coker}\Gamma_\xi^{n-1}, & \textit{if } \Theta_\xi^{n-1} \textit{ is monomorphic}, \end{cases}$$

where Θ_ξ^i, Γ_ξ^i are the homomorphisms of (1.1).

This is a generalization of a part of the theorem of I. M. James [8, Th. 5.1]