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Isomorphisms between Interval Sublattices of an Orthomodular Lattice

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1. Introduction.

This paper deals with the following question: given orthogonal projective elements a, b of an orthomodular lattice $\mathscr{L}$, under what general circumstances are the interval sublattices $\mathscr{L}(0, a)$, $\mathscr{L}(0, b)$ orthoisomorphic?

The answer that we offer provides a description of a class of lattices, called uniform, in which not only do the indicated orthoisomorphisms exist, but they are explicitly displayed as simple lattice polynomials. This desirable state of affairs is achieved through the use of a strong postulate that requires the existence of certain special kinds of elements of $\mathscr{L}$.

The postulate is framed in terms of a new relation "UA" between pairs of non-zero elements p, q of an orthomodular lattice $\mathscr{L}$. We write $p \sharp q$ when

$$x \leq q \Rightarrow (p \wedge (p^{\perp} \vee x)) \perp (q \wedge x^{\perp}).$$

This resembles the condition that p, q form a modular pair, and is in fact stronger (see (4) of 2.4 and remarks following the proof of 4.1). The relation UA (p, q) is then defined as the symmetrization of $\sharp$, subject to a side condition to rule out trivial complications. The exact definition is this: UA$(p, q) \Leftrightarrow$ both $p \sharp q$, $q \sharp p$ and $p \wedge q = p \wedge q^{\perp} = p^{\perp} \wedge q = 0$. The letters UA are intended to suggest "uniform angle", and the relationship UA(p, q) may be read as "p and q have a uniform angle between them". This terminology is derived from a geometric interpretation available when $\mathscr{L}$ is the lattice of projections of Hilbert space — see 4.4.

A uniform orthomodular lattice is defined by the following property: given any pair of non-zero orthogonal projective elements a, b, there is an element $h \leq a \oplus b$ that makes a uniform angle with both a and b. We call such an element h "splitting" for the pair a, b. The desired orthoisomorphism between the interval sublattices $\mathscr{L}(0, a)$, $\mathscr{L}(0, b)$ is constructed through the use of the special properties of the splitting element h.

This definition has the advantage of being easily verified in a large class of examples, namely the projection lattices of von Neumann algebras, and does lead swiftly to a simple, explicit formula for the desired orthoisomorphisms (Theorem 3.1). Another possible advantage is that the explicit nature of the definition may promote the building of a reasonably detailed theory of these lattices.