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Harmonic Functions and the Borel-Weil Theorem

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§ 1. Introduction

In the previous paper [3], we proved that, for non-zero eigenvalues, arbitrary eigenfunctions of the laplacian can be given by the "Poisson integral" of elements of a certain space $\tilde{\mathscr{B}}(S^{n-1})$ which contains the space of hyperfunctions on the $(n-1)$ dimensional unit sphere as a proper subspace.

In case the eigenvalue is zero, however, the Poisson integral gives only constant functions.

In this paper, we shall give the modification of the Poisson integral so that, using the Borel-Weil theorem, the modified "Poisson integral" gives the canonical isomorphism between the space of all homogeneous harmonic polynomials on $\boldsymbol{R}^n$ of degree m and the space of all holomorphic sections of a certain $SO(n, \boldsymbol{C})$ -homogeneous holomorphic line bundle L_m over the Grassmann manifold $SO(n)/SO(2)\times SO(n-2)$. In the last section, we shall consider a certain space $\oplus\sum_{m\geq 0}\Gamma(L_m)$ and show that every harmonic function on $\boldsymbol{R}^n$ can be represented by an analogue of the "Poisson integral" of the unique element of $\oplus\sum_{m\geq 0}\Gamma(L_m)$.

§ 2. Homogeneous harmonic polynomials

In this section we shall refer to general properties about harmonic polynomials which we need in the following sections. In this paper, we denote by G the rotation group of degree n, where n is a positive integer. For each non-negative integer m, let $\mathscr{H}^{n,m}$ denote the space of all homogeneous harmonic polynomials on $\boldsymbol{R}^n$ of degree m. By left translations, one obtains an irreducible (unitary) representation τ_m of G on $\mathscr{H}^{n,m}$. The representation τ_m is of class one with respect to the subgroup H' of G consisting of all elements of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix} \quad : \quad h\in SO(n-1, \boldsymbol{R}),$$

and every irreducible representation of G of class one with respect to H' is equivalent to τ_m for some non-negative integer m.

Let P^n be the ring of polynomial function on $\boldsymbol{R}^n$ with coefficients in the complex field $\boldsymbol{C}$, and $P^{n,m}$ be the subspace of P^n consisting of all m-homogeneous